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REINSURANCE MARKET MECHANISMS AND DIVIDEND STRATEGIES
FOR AN INSURANCE COMPANY

Pantelis M. Pechlivanides

California University

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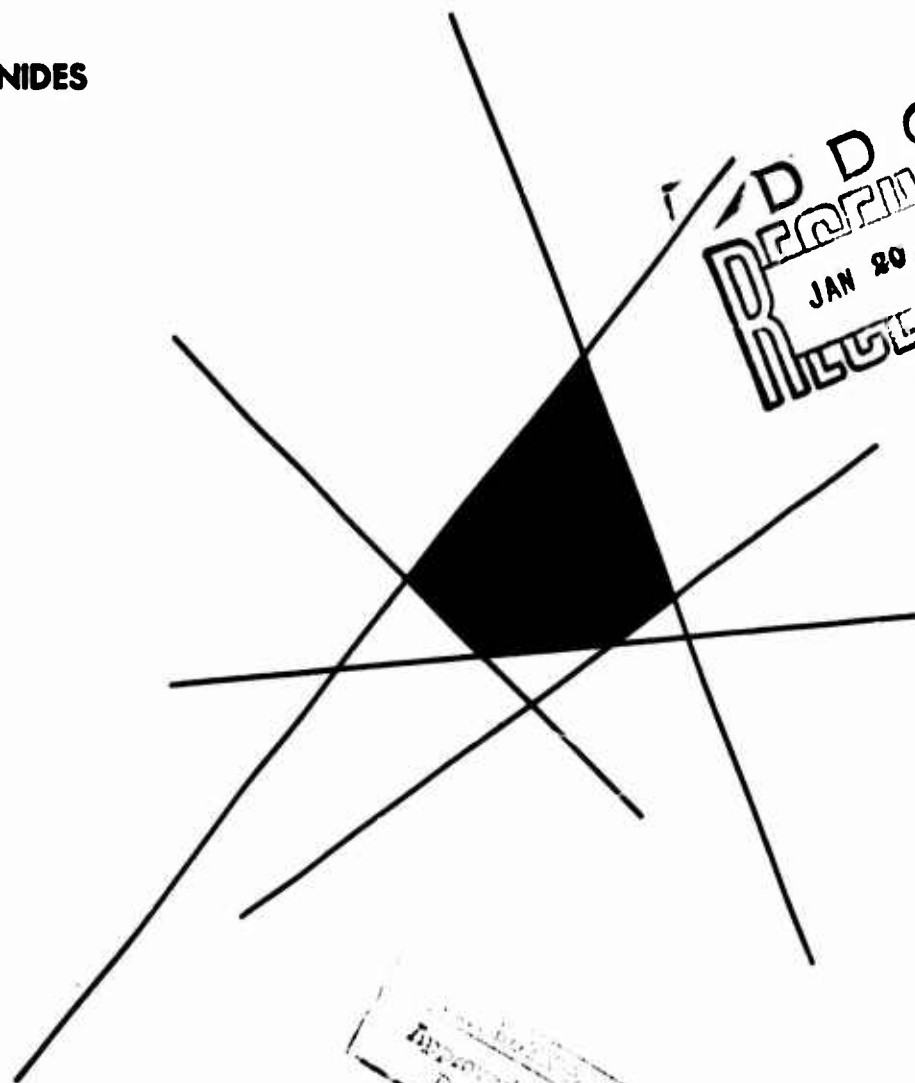
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by
PANTELOS M. PECHLIVANIDES

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ABSTRACT

This study consists of two parts:

- (1) A Reinsurance Market is described and studied in a manner similar to those of stock market theories and the existence of an equilibrium price function is shown.
- (2) The sequential reinsurance-dividend problem of the Insurance Company (I.C.) is formulated as a dynamic programming problem and closed form solutions found for a class of utility functions. It is shown that the optimal dividend strategy is linear in the reserves level and that the prereinsurance wealth of the I.C. does not influence the optimal *form* of its postreinsurance wealth.

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CHAPTER 1

INTRODUCTION

One of the classic problems of an insurance company (I.C.) is control of its *level of risk reserves*. These reserves, as we shall henceforth call them, are loosely defined as a sum of money to which *premiums*, collected by selling policies, are added and *losses*, amounts paid to satisfy claims, are subtracted over time. The management of an I.C. can control the reserves process in four ways: (1) By changing the income stream (by increasing the premiums or the volume of business); (2) By influencing the loss amounts (through safety inspections or improved underwriting); (3) By modifying the form of the loss distribution to the I.C. (by buying (selling) *re-insurance*); (4) By paying *dividends* to the policyholders or shareholders. In this thesis we shall at first assume that the first two factors (rate making, marketing and underwriting) are kept constant and we shall examine in detail the interaction of reinsurance and dividend decisions. It will then be shown how a generalization can be made to partially include the first factor through a decision for expenditures to promote sales.

By reinsuring, the claims random variable is transformed; in other words, the I.C. insures a portion of its random loss by making a fixed payment to another I.C. The first I.C. is called the *cedent* and the second the *reinsurer*. The most common forms of reinsurance treaties (or contracts) are the *stop-loss* and *quota reinsurance*. In the former, the reinsurer promises to pay the claims exceeding a certain limit (much like the *deductible* on the customer - I.C. level), while in the latter case, the reinsurer promises to

pay a fixed quota of the total claims. Other treaties are of course possible. In more complicated situations, two or more I.C.'s may mutually reinsure one another in an attempt to spread the risk.

By paying dividends the I.C. can, of course reduce its reserves at any time. This gives an immediate benefit to policy/shareholders, but then it will be unable to bear a lot of risk without an increased probability of being ruined which translates to less expected profits or dividends in the future.

Most of the literature treats the two above-mentioned ways of controlling the reserves separately.

Dividend Models

Models for optimal dividend payment strategies were initiated by De Finetti (1957), who sought a criterion for evaluating the reserves considered as a stochastic process. He regarded the criterion used up to that time (the probability of ruin, i.e. the probability that the reserves would become zero or negative) as being too conservative, and proposed using the discounted sum of dividend payments. In his model, every time the reserves exceeded a certain level H , the excess was distributed as dividend. The problem was then to find the optimal H so as to maximize the discounted sum of dividend payments until ruin occurred.

This kind of *barrier strategy* was later proven to be optimal for the discounted sum of dividends criterion [Miyasawa (1962), Morrill (1966)]. Borch (1966), (1968a) found solutions to the problem for some nondiscrete distributions and was also able to calculate the expected time until ruin under a barrier strategy. Gerber (1969) extended the results to the continuous time case. He proved the

optimality of the barrier strategy and calculated the optimal barrier level when the claims process was compound Poisson.

Reinsurance Models

Paralleled in time, but with a much longer history, was the development of reinsurance models which can be divided into two classes. To the first class belong all the models where the *form* of the reinsurance contract is given explicitly (for example, *stop-loss* or *quota*). Questions usually associated with these models are: What is the fair net premium of reinsurance?, Is there any benefit (e.g. reduction of variance or of the probability of ruin) to the cedent or the reinsurer or both?, What is the optimum level of reinsurance etc?

The second class of reinsurance models, developed relatively recently, focuses on finding the *optimal forms* of reinsurance under different preference criteria for the cedent, the reinsurer, or both. A brief review of this second class of models follows, since they are closely related to this study.

Considering reinsurance from the cedent's point of view, Borch (1960c) proved that the stop-loss form of treaties is optimal when the cedent seeks to minimize his variance. Kahn (1961) generalized this result when the cedent is permitted to choose from a wider class of contracts which he called admissible. Introducing the expected utility criterion, Arrow (1963) showed that if the premium associated with an insurance (or reinsurance) contract depends only on its actuarially fair value (i.e. the expected value of the contract), then stop-loss reinsurance is optimal for the cedent. Several other authors considered the problem from the expected utility maximization

point of view and found conditions for full or partial coverage under stop-loss or quota insurance (reinsurance). [See, for example, Pashigian, Schkade, Menefee (1966), Smith (1968), Mossin (1968)].

Looking at the problem from the reinsurer's point of view, Vajda (1962) showed that quota reinsurance is preferred by the reinsurer who seeks to minimize his coefficient of variation.

Ohlin (1969) generalized existing work by considering various classes of contracts and finding the best contract in each class independently for the cedent and the reinsurer. Similar results were found by Lemaire (1973) who used a rather unorthodox preference rule.

The most important model as related to this thesis is that of Borch (1960a), (1960b) who tried to find a "best" treaty for both cedent and reinsurance. Borch viewed the situation as an n -person cooperative game and found conditions for a reinsurance treaty to be Pareto optimal. This approach is intuitively more satisfying as it considers both sides of the problem simultaneously. We will expand on this model in detail in Chapter 2.

Combined Models

Three questions arise naturally at this point:

Q1: How can we combine the dividend payment decision with the reinsurance problem in a single dynamic decision formulation?

Q2: How does this combined decision influence the form of reinsurance to be chosen?

Q3: If it is possible to solve Q1 and Q2 from the I.C.'s point of view, what do we lose, if anything, by disregarding the other parties that take part in the reinsurance contract? In other words,

is it possible to solve the individualistic problem of the firm without sacrificing the cooperative nature of reinsurance as described by Borch (1960a), (1960b)?

As far as Q1 is concerned, there are two models in the literature:

A. Dayananda (1970), (1972) assumes a continuous time, infinite horizon model of an I.C. whose losses form a Wiener stochastic process. He further assumes that the I.C. is restricted to quota reinsurance and that its dividend policy is to distribute to the shareholders the excess of its reserves above a fixed level H . Using as the objective function a heuristic form of the expected dividend payments, he finds the optimal reinsurance quota and the barrier level H . In particular, he determines the conditions for which no reinsurance, quota reinsurance, or a mixture of the two (according to the level of reserves) is optimal.

B. In contrast, André Frisque (1974) formulates a discrete time dynamic programming model where, in each period, the I.C. chooses both the amount to be paid as dividend and the portion of the portfolio to be reinsured, given that the I.C. is restricted to quota reinsurance and the price of reinsurance is actuarially fair. The objective of the I.C. is to maximize the expected discounted sum of the one-period utility functions. A closed form solution is found when: (a) the one-period utility function is of the form $u(x) = x^\gamma$ $x > 0$, $0 < \gamma < 1$; (b) the claims variable can take the values 0 or 2; and (c) the premiums received from policyholders is 1. Both the optimal dividend payment policy and the optimal quota reinsurance level are found to be linear in the reserves.

Our Approach

The last model is very much related to our approach and can be considered as a special case of the results of Chapter 3. We, shall in fact, find closed-form solutions for a much wider class of utility functions (the linear risk tolerance class to be defined in detail in Chapter 3), while imposing no restrictions on either the form of reinsurance or the probabilistic structure of claims. Both our approach and that of Frisque (1974) are related to the consumption-investment model of the individual, studied by Hakansson (1966).

Both of the above models (Dayananda (1970), (1972), Frisque (1974)) do not address to questions Q2 and Q3. This study attempts to answer these questions as related to the dynamic decision problem of the I.C.

In Chapter 2 we review the theory on Pareto optimal reinsurance treaties, and use these results to construct a mutual reinsurance market of n I.C.'s (agents) where each tries to maximize its expected utility over all possible reinsurance contracts, when prices are given. We then show, using a fixed point theorem, that the market has an equilibrium (i.e. a prices exist that clear the market) under certain conditions. This result implies that the competitive market equilibrium is also Pareto optimal, answering question Q3, and defining an equilibrium price function for the market.

Chapter 3 considers the individual firm's dynamic decision problem, when faced with reinsurance price function in each period. We formulate a dynamic programming relation which, in each period, allow the I.C. to find the best reinsurance contract and the dividend to

be paid to the policy-share-holders when the objective is to maximize the expected value of the discounted sum of the one-period utilities. Closed form solutions are found for the linear risk-tolerance class of utility functions. In this manner Q2 and Q3 are answered.

In Chapter 4 we interpret the results of Chapter 3 and generalize the model to include a decision on expenditure for promotion of sales, which, in fact, does not alter the nature of the optimal policies.

In Chapter 5, we consider a multiplicative utility function [Meyer (1969)], in contrast to the discounted sum of utilities used in Chapter 3. Closed form solutions are found only for a subclass of the linear-risk-tolerance class of utility functions, which, nevertheless, strikingly resemble the results of Chapter 3.

Finally in Chapter 6 we discuss some further questions and possible extensions of this study.

CHAPTER 2

THE MUTUAL REINSURANCE MARKET

2.1 Introduction

We review in Section 2.2 the Pareto optimal mutual reinsurance treaty conditions found by Borch (1960a), (1960b). This approach considers the participating companies as fully cooperating in the sharing of risk. His solution allows a good deal of freedom, as in general an infinity of Pareto optimal treaties is determined. In the next Section (2.3), we consider a competitive situation which we call the Mutual Reinsurance Market. In the market there are n agents who exchange "elementary" contracts under given prices each with the purpose of maximizing his expected utility function under a budget constraint. Conditions for the optimality of each agent's maximization problem are formulated in a manner similar to recent securities market models. [See Borch (1967b).] These conditions show that the results of the market will belong to a set of Pareto optimal treaties if, in addition, the market clearing condition is satisfied. In Section (2.4) we use a fixed point theorem to prove that this is possible if certain conditions are satisfied. Finally, in Section (2.5) three simple examples are shown to clarify the already exposed theory.

2.2 The Pareto Optimal Reinsurance Treaty2.2.1 General

Consider n insurance companies (agents). Each agent i possesses a capital C_i (reserves plus premiums collected) and a portfolio of contracts whose total claims are given by the random variable

ξ_1 . Let $\underline{\xi} \equiv \xi_1, \dots, \xi_n$. Let X_i be the set of possible values of ξ_i ($i = 1, \dots, n$) and $X \equiv X_1 \times X_2 \times X_3 \times \dots \times X_n$.

For simplicity we will denote $\psi_1 \equiv C_1 - \xi_1$. ψ_1 is a random variable whose value will be denoted y_1 ; it may be interpreted as the net end-of-period capital when no reinsurance is purchased.

Also let $\underline{\psi} \equiv \psi_1, \dots, \psi_n$. The value of $\underline{\psi}$ will be \underline{y} ; $y_i \in Y_i$ and $\underline{y} \in Y \equiv Y_1 \times \dots \times Y_n$.

A reinsurance treaty among the n agent is a n -triple of functions $Z(\underline{y}) = \{Z_1(\underline{y}), \dots, Z_n(\underline{y})\}$ such that

$$(2.1) \quad \sum_{i=1}^n Z_i(\underline{y}) = \sum_{i=1}^n y_i \quad \text{for all } \underline{y} \in Y.$$

In other words, each agent, prior to reinsurance, has the risky asset ψ_1 . The reinsurance treaty redistributes the risks, thus assigning to each agent i a function (or contract) $Z_i(\underline{y})$ describing his wealth as a function of the realization of the random variables $\underline{\psi}$ (equivalently the claims, ξ_i 's) so that there is conservation (2.1).

Each agent is further characterized by a utility of money, $u_i(\cdot)$. Then the reinsurance treaty changes the agent's utility from

$$U_1(\psi_1) \equiv Eu_1(\psi_1)$$

to

$$U_1(Z_1(\underline{\psi})) \equiv Eu_1(Z_1(\underline{\psi}))$$

where expectation is taken with respect to the joint probability distribution of $\underline{\psi} = \psi_1, \dots, \psi_n$.

A reinsurance treaty $Z_1(\underline{y}), \dots, Z_n(\underline{y})$ is called *Pareto optimal* if for every other reinsurance treaty $Z'_1(\underline{y}), \dots, Z'_n(\underline{y})$

$$\text{either } U_i(Z'_1(\underline{\psi})) = U_i(Z_1(\underline{\psi})) \text{ for all } i$$

$$(2.2) \quad \text{or for at least one } i$$

$$U_i(Z'_1(\underline{\psi})) < U_i(Z_1(\underline{\psi})) .$$

There are alternative ways of expressing the Pareto optimal concept but the idea is the same. The set of Pareto optimal reinsurance treaties includes all those treaties for which the utility of one or more agents cannot be increased without decreasing the utility of some other agent(s).

If we assume that reinsurance is conducted in a cooperative mood, it is reasonable to assume that the participating agents will try to achieve a Pareto optimal result. (The philosophy is: "Why not make a treaty that will increase the utility of all participants".) This, however, is not always the case. In reality, it is possible that a company might wish to suffer a slight loss if that meant a heavy loss or bankruptcy for its competitor. We will, however, disregard such situations here and adopt the Pareto optimal approach instead.

2.2.2 The Theory

Borch (1960a), (1960b) presented necessary and sufficient conditions for the Pareto optimality of an n -agent reinsurance treaty. Later, Du Mouchel (1968), provided a more rigorous proof along with conditions for the existence of a Pareto optimal solution. The following theorem summarizes these results.

Theorem 1:

Let the utility functions $u_i(\cdot)$, $i = 1, \dots, n$ be differentiable, nondecreasing, concave. Then the reinsurance treaty $\{Z_1(y), \dots, Z_n(y)\}$ is Pareto optimal if and only if there exist nonnegative constants $k_1 \neq 0, k_2, \dots, k_n$ so that

$$u'_i(Z_i(y)) = \frac{k_1}{k_i} u'_1(Z_1(y)) \quad i = 1, \dots, n, \forall y \in Y$$

(2.3)

$$\text{and} \quad \sum_{i=1}^n Z_i(y) = \sum_{i=1}^n y_i.$$

Furthermore, if the $u'_i(\cdot)$ are all continuous and the k_i are so chosen so that the ranges of the functions $k_i^{-1} u'_i(\cdot)$ have a common nonempty intersection, then a Pareto optimal reinsurance treaty exists which satisfies the conditions (2.3).

Proof:

The proof can be found in the papers by Borch and Du Mouchel mentioned above. Here we will outline the sufficiency part of the proof, i.e. the fact that (2.3) implies Pareto optimality.

Following Du Mouchel (1968), let $\{Z_i(y)\}_{i=1}^n$ satisfy (2.3) for some positive constants k_2, \dots, k_n .

The case when a $k_i = 0$ is straightforward, as for agent i $u_i(\cdot)$ has reached a maximum, and the treaty $\{Z_i\}_{i=1}^n$ cannot be improved from his point of view. So $\{Z_i\}_{i=1}^n$ is Pareto optimal.

We consider, therefore, the case when $k_i > 0$ $i = 1, \dots, n$, $k_1 = 1$. Form any other treaty

$$Z'_i = Z_i + e_i, \quad e_i = e_i(\psi), \quad i = 1, \dots, n.$$

Then because $\sum z'_1 = \sum z_1 \Rightarrow \sum_{i=1}^n e_i = 0$.

Now for each i

$$U_i(z'_1) - U_i(z_1) = E[U_i(z'_1) - u_i(z_1)]$$

but since u_i is concave,

$$u_i(z_1 + e_i) - u_i(z_1) \leq u'_i(z_1)e_i$$

$$U_i(z'_1) - U_i(z_1) \leq Eu'_i(z_1)e_i$$

and because of (2.3)

$$U_i(z'_1) - U_i(z_1) \leq e_i k_i Eu'_i(z_1).$$

Dividing by k_i and summing over i ,

$$(2.4) \quad \frac{\sum [U_i(z'_1) - U_i(z_1)]}{k_i} \leq Eu'_i(z_1) \left(\sum e_i \right) = 0.$$

If z_1 were not Pareto optimal then there would be a z'_1 such that for each i $U_i(z'_1) \geq U_i(z_1)$ and for at least one j $U_j(z'_1) > U_j(z_1)$, but this contradicts (2.4) above. Thus $\{z_1\}$ is Pareto optimal.

We omit the necessary and existence parts of the proof. ||

2.2.3 Observations

1) We implicitly assumed that all agents agree on the probability distribution of ψ . If this were not the case, we could still prove a similar result, the necessary and sufficient conditions for Pareto optimality now being

$$(2.5) \quad u'_1(z_1(y))\phi_1(y) = \frac{k_1}{k_1} u'_1(z_1(y))\phi_1(y)$$

$$i = 1, \dots, n, \forall y \in Y$$

where $\phi_1(y)$ is the *subjective probability density* function of agent i for the random variable ψ . But we must further assume that all agents have $\phi_1(y) > 0 \quad y \in Y$ and $\phi_1(y) = 0 \quad y \notin Y$. (i.e. agents agree on the set of values ψ can take).

Also in this case, the existence part of the Theorem does not hold anymore. Nevertheless, if the $u'_1(\cdot)$ have a range of $(0, \infty)$ and in addition to being continuous they are also strictly decreasing, a solution exists for all $k_1 > 0 \quad i = 1, \dots, n$. Here is a proof:

Since $u'_1(\cdot)$ have a range of $(0, \infty)$ and $\phi_1(y) > 0$ for $y \in Y$ for all agents, (2.5) has a solution. It only remains to show that the conservation condition

$$(2.1) \quad \sum_{i=1}^n z_i(y) = \sum y_i \equiv y^+$$

is satisfied.

$$\text{From (2.5) } z_1(s) = u_1^{-1}\left(k_1 \frac{s}{\phi_1}\right).$$

Thus we have to check if s can be chosen as a function of y so that $V(s) \equiv \sum u_1^{-1}\left(k_1 \frac{s}{\phi_1}\right)$ equals y for all y . Indeed each $u_1^{-1}(\cdot)$ is continuous, thus $V(s)$ is continuous in s . Also the range of $u_1^{-1}(\cdot)$ is, for example, $(\underline{t}_1, \bar{t}_1)$. Now if we require

[†]For completeness, we note that the problem is mathematically the same if $\sum y_i$ is replaced by a function $N(y)$. If we think then of ψ as signifying the "state of nature," we get into more general situations which include the securities market, for which results like (2.5) are also known to apply [see for example Mossin (1973a), (1973b)].

that $\int \underline{t}_1 \leq y \leq \int \bar{t}_1$ (which must hold, otherwise the pre-reinsurance situation would have no meaning) it follows that by letting $s \rightarrow 0$, $V(s) \rightarrow \int \bar{t}_1$, while for $s \rightarrow \infty$, $V(s) \rightarrow \int \underline{t}_1$. Thus $V(s)$ covers the range of y and is continuous. This means that $s(y)$ can be found so that the conservation condition (2.1) is satisfied.

2) When we assume that $\phi_i(y) = \phi(y)$, $i = 1, \dots, n$ as in (2.3), we can prove that the $Z_i(y)$, $i = 1, \dots, n$ depend only on

$y \equiv \sum_{i=1}^n y_i$ and not on each particular y_i . In other words, when

the insurance companies agree on the probability distribution of the risks, then they pool their risks and the reinsurance contracts depend on the pooled risk (ψ) only. To avoid new notation we will simply write $Z_i(y)$. The proof by Borch (1960b) is as follows:

By Theorem 1 a Pareto optimal reinsurance treaty $\{Z_i\}$ must satisfy

$$(2.3) \quad \left. \begin{aligned} u'_i(Z_i(y)) &= k_i u'_1(Z_1(y)) \\ \sum_{i=1}^n Z_i &= \sum_{i=1}^n y_i \end{aligned} \right\} \begin{aligned} i &= 2, \dots, n \\ k_i &\geq 0 \end{aligned}$$

Differentiating both equations w.r.t. y_j we have,

$$\frac{\partial Z_i(y)}{\partial y_j} u''_i(Z_i) = k_i \frac{\partial Z_1(y)}{\partial y_j} u''_1(Z_1)$$

$$\sum_{i=1}^n \frac{\partial Z_i}{\partial y_j} = 1.$$

Assume that $u_1''(\cdot) \neq 0$ (in fact < 0), we divide the first equation by $u_1''(Z_1)$ and sum over i :

$$1 = u_1''(Z_1) \frac{\partial Z_1}{\partial y_j} \sum_{i=1}^n \frac{k_i}{u_i''(Z_i)}.$$

It follows that

$$\frac{\partial Z_1}{\partial y_j} = \frac{k_j}{u_1''(Z_1)} \cdot \frac{1}{\sum_{h=1}^n \frac{k_h}{u_h''(Z_h)}}.$$

Since the right hand side does not depend on j , it follows that

Z_1 is a function of $y \equiv \sum_{j=1}^n y_j$ only.

3) Equations (2.3) give us necessary and sufficient conditions for a reinsurance treaty to be Pareto optimal. We must realize, however, that in general there are going to be an infinity of Pareto optimal solutions for each choice of $\underline{k} \equiv k_1, \dots, k_n$. (We have set $k_1 = 1$ since the solution brought about by \underline{k} is the same as that of $a \cdot \underline{k}$, $a > 0$). In general, the range of \underline{k} will be limited as the existence part of Theorem 1 requires. If, however, the $u_i'(\cdot)$ have a range of $(0, \infty)$ and are strictly decreasing and continuous, then a Pareto optimal solution is guaranteed for all $k_i > 0$, $i = 1, \dots, n$ even if heterogeneous probabilities are introduced as discussed above (Section 2.2.3/1).

4) An equivalent problem to solving (2.3) is

Find $P(\underline{y})$, $Z_1(\underline{y})$, ..., $Z_n(\underline{y})$ such that

$$u'_i(Z_i(\underline{y})) = k_i \frac{P(\underline{y})}{\phi(\underline{y})} \quad i = 1, \dots, n, \quad k_i > 0, \quad k_i \geq 0$$

$$i = 2, \dots, n$$

(2.3)'

$$\sum_{i=1}^n Z_i(\underline{y}) = y$$

where $\phi(\underline{y})$ is the probability density function of $\underline{\psi}$. Let us also observe that since $u'_i(\cdot) \geq 0$ and $\phi(\underline{y}) > 0$, $P(\underline{y}) \geq 0$ ($u'_i(\cdot) > 0 \Rightarrow P(\underline{y}) > 0$).

The solution to (2.3)' will be a function of $\underline{k} = k_1, \dots, k_n$ as well as \underline{y} . We can then write

$$P(\underline{y}) = P(\underline{y}, \underline{k}), \quad Z_i(\underline{y}) = Z_i(\underline{y}, \underline{k}), \quad i = 1, \dots, n$$

for all \underline{k} so that a solution to (2.3)' exists. By observation of (2.3)' we see that for $a > 0$

$$P(\underline{y}, a \cdot \underline{k}) \equiv \frac{1}{a} P(\underline{y}, \underline{k}), \quad Z_i(\underline{y}, a \cdot \underline{k}) \equiv Z_i(\underline{y}, \underline{k}), \quad i = 1, \dots, n.$$

5) If $\phi_i(\underline{y}) = \phi(\underline{y})$, $i = 1, \dots, n$ then $\frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})}$ is decreasing in each y_i if each $u'_i(\cdot)$ is nondecreasing.

We will use this result in Chapter 3.

The proof simply consists in substituting

$$Z_i = u_i^{-1} \left(k_i \frac{P}{\phi} \right) \quad \text{in} \quad \sum Z_i = y$$

and noting that since each $u_i^{-1}(\cdot)$ is nonincreasing an increase

in y should induce a decrease in $\frac{P(y, \underline{k})}{\phi(y)}$.

2.2.4 Example - The Quadratic Case

Let $u_1(x) = -\frac{1}{2}x^2 + \beta_1 x$. Then $u'_1(x) = -x + \beta_1 > 0$ and decreasing if $\beta_1 > 0$ and $x \leq \beta_1$. Substituting in (2.3)

$$-Z_1 + \beta_1 = k_1(-Z_1 + \beta_1) \quad i = 2, \dots, n.$$

Summing over i and using $\sum_{i=1}^n Z_i = y$

$$-y + \sum \beta_i = \sum_{i=1}^n k_i(-Z_1 + \beta_1)$$

or in general

$$Z_j(y, \underline{k}) = \frac{k_j}{\sum_{i=1}^n k_i} \left(y - \sum \beta_i \right) + \beta_j$$

which as expected, depends on y only. (Recall Section 2.2.3/2.)

Note also that $y - \sum \beta_i \leq 0$ (since $y_i \leq \beta_i$) so $Z_j \leq \beta_j$ as it should. This treaty is an example of a quota reinsurance treaty as it is linear in the aggregate after-claims wealth of the participating companies.

The result is valid for all $k_i \geq 0$ $i = 2, \dots, n$; $k_1 = 1$.

Further, from (2.3)'

$$\frac{P(y, \underline{k})}{\phi(y)} = \frac{1}{k_1} u'_1(Z_1(y, \underline{k})) , \quad k_1 \neq 0$$

$$= \frac{1}{k_1} (-Z_1 + \beta_1) = \frac{\sum \beta_i y_i}{\sum k_i} ,$$

and $P(y, k) \geq 0$ as expected.

2.3 The Reinsurance Market

We discussed in the last section the conditions that must be satisfied by a reinsurance treaty among n agents for the result to be Pareto optimal. The question we consider now is whether a market mechanism can be formed that can achieve a Pareto optimal result. We will first define the commodities (or elementary contracts) to be traded and find the conditions for optimality of each agent when he tries to maximize his expected utility subject to a budget constraint when prices for the elementary contracts are given. Then in Section 2.4 we will show that if certain Assumptions hold the allocation that results from trading elementary contracts belongs to the Pareto optimal allocations. The approach and results resemble that of the securities market theories. [See for example Borch (1968b), Mossin (1973a).]

2.3.1 The Market Optimality Conditions

Let us look at the space Y where y , the value of the random variable ψ , is defined, and denote by $L_2(Y)$ the set of square integrable functions defined on Y . By the theory of Hilbert spaces, there exists a set of functions in $L_2(Y)$ called a *complete set* such that all functions in $L_2(Y)$ can be written as a linear combination of the elements of the complete set. Further, if the elements of the complete set of functions are orthonormalized, the set is called an *orthonormal basis* for $L_2(Y)$ [see for example Reed and Simon (1972), p. 44].

Let $\{g_j(y)\}_{j=0}^{\infty}$ be a basis for $L_2(Y)$. Then for a square

integrable $Z_i(\underline{y})$ we can write:

$$(2.5) \quad Z_i(\underline{y}) = \sum_{j=0}^{\infty} e_j^i g_j(\underline{y}) \quad i = 1, \dots, n.$$

where

$$e_j^i \equiv \int_Y Z_i(\underline{y}) g_j(\underline{y}) d\underline{y} \quad \begin{matrix} (i = 1, 2, \dots, n) \\ (j = 1, 2, \dots, \infty) \end{matrix}$$

are the coefficients in the expansion of the i^{th} reinsurance treaty.

$d\underline{y} \equiv dy_1, dy_2, \dots, dy_n$.

Assume for the moment that Y is such that each y_1 varies over a finite interval. Then, with the same reasoning, we can also expand the net prereinsurance wealth of the i^{th} agents using the same basis as:

$$(2.6) \quad y_1 = \sum_{j=0}^{\infty} f_j^1 g_j(\underline{y}).$$

We can now return to our problem and make the following reinterpretations:

- ψ_i : the prereinsurance wealth of agent i , a random variable whose value is y_1
- $Z_i(\underline{y})$: the postreinsurance wealth (or contract) of agent i when $\underline{\psi} = \underline{y}$
- $g_j(\underline{y})$: an elementary contract of type j that pays \$ $g_j(\underline{y})$ when $\underline{\psi} = \underline{y}$
- f_j^1 : quantity of $g_j(\underline{\psi})$ that agent i possesses, prior to reinsurance.
- e_j^1 : quantity of $g_j(\underline{\psi})$ that agent i possesses after reinsurance.

This interpretation suggests that the commodities traded were the elementary contracts $g_j(\underline{y})$ ($j = 0, 1, 2, \dots$), a linear combination of which can form any "real" reinsurance treaty $Z(\underline{y})$ within $L_2(Y)$.

Suppose that it were possible to specify a price p_j for each $g_j(\underline{y})$, ($j = 0, 1, 2, \dots$). Let $\sum_{j=0}^{\infty} p_j^2 < \infty$.

Each agent in the market tries to maximize his own expected utility subject to his budget constraint, given the prices. Thus the maximization problem of the i^{th} agent in the market can be restated as:

$$(2.7) \quad \left\{ \begin{array}{l} \max_{\{e_j\}_{j=0}^{\infty}} Eu_i \left(\sum_{j=0}^{\infty} e_j^i g_j(\underline{y}) \right) \\ \text{s.t.} \quad \sum_{j=0}^{\infty} e_j^i p_j = \sum_{j=0}^{\infty} \bar{e}_j^i p_j \end{array} \right.$$

(2.7) can be solved directly, but was introduced only to explain clearly what basic commodities were being traded in the market. We will now give the problem an equivalent form that is easier to solve and also is more compact as we avoid expansions of functions in an orthonormal basis.

Define a price function $P(\underline{y})$ as:

$$P(\underline{y}) \equiv \sum_{j=0}^{\infty} p_j g_j(\underline{y}) .$$

Then the i^{th} agent's maximization problem in a market characterized by a price function $P(\underline{y})$ is written as:

$$\begin{aligned}
 (2.7)' \quad & \max_{Z_1(\psi)} Eu_1(Z_1(\psi)) \\
 & \text{s.t.} \quad \int_Y Z_1(y)P(y)dy = \int_Y y_1P(y)dy .
 \end{aligned}$$

That is, the i^{th} agent seeks to maximize his expected utility over all possible reinsurance contracts $Z_1(\cdot)$ subject to his budget constraint.

Assuming that $u_1(\cdot)$ is differentiable, nondecreasing, concave, the necessary and sufficient conditions for $Z_1(y)$ to be an optimal solution to (2.7)' is given by the calculus of variations. [See for example Gelfand and Fomin (1963)] as:

$$(2.8) \quad u'_1(Z_1(y))\phi(y) = \lambda_1 P(y), \forall y \in Y$$

and

$$(2.9) \quad \int_Y Z_1 P dy = \int_Y y_1 P dy$$

where λ_1 is a Lagrange multiplier to be determined by (2.9).

Equations (2.8) and (2.9), which must hold for all agents $i = 1, \dots, n$, are called the *Market Optimality Conditions*.

Similar equations appear as optimality conditions in a securities market.[†] This is to be expected since the elementary contracts $\{g_j(\psi)\}$ could easily be regarded as securities promising to pay $g_j(y)$ if the "state of nature" $\psi = y$ occurs. The reason that, in our case, the optimality conditions take a rather simple form (2.8), (2.9) is that we assume that the securities (or elementary contracts) $\{g_j(\psi)\}$ form a complete orthonormal set.

[†] See for example Mossin (1973a), (1973b).

Arrow Certificates - an Example

A particular case would be to consider Y as consisting of the integers from 1 to m (i.e. there are m states of nature). Then any function over Y can be written as a linear combination of the impulse functions,

$$\delta_j(y) = \begin{cases} 1 & y = j \\ 0 & \text{otherwise} \end{cases} \quad j = 1, \dots, m.$$

$\delta_j(y)$ is a certificate promising to pay \$1 if $y = j$ and nothing otherwise. These are known as "Arrow certificates", first used by Arrow (1964) to introduce uncertainty in equilibrium models of economics. In this space, therefore, completeness means that there is a certificate for each state of nature. In the more general sense, completeness means that there are enough elementary contracts so that any desired contract can be formed as a linear combination of them.

2.3.2 Some Remarks on the Market Optimality Conditions

1) Let $u'_1(\cdot) > 0$. If the Market Optimality Condition (2.8) is to make sense, we require that $\lambda_1 P(\underline{y}) > 0$. Thus $P(\underline{y})$ must be positive for all \underline{y} or < 0 for all \underline{y} . This is a matter of definition. Consider $P(\underline{y})d\underline{y}$; it is the price of a certificate that pays \$1 if $\underline{y} \in (\underline{y}, \underline{y} + d\underline{y}]$, zero if otherwise. It makes sense then to ask that $P(\underline{y}) > 0$. It follows that $\lambda_1 > 0$.

2) Let $\pi \equiv \int P(\underline{y})d\underline{y}$. Then π is the price of a contract that pays \$1 under any circumstances. It follows that $i = \frac{1-\pi}{\pi}$ is the interest rate. If $i > 0 \Rightarrow \pi < 1$.

3) If the $u'_1(\cdot)$ have a range of $(0, \infty)$ then (2.8) has a solution, $Z_1(\underline{y}, \lambda_1)$ for any $\lambda_1 > 0$. Further, if $u'_1(\cdot)$ is strictly decreasing and continuous, then $Z_1(\underline{y}, \lambda_1)$ is continuous and strictly decreasing in λ_1 . Furthermore, a solution λ_1 to (2.9) always exists and thus the Market Optimality Conditions have a solution. To see the last assertion, let the domain of $u'_1(\cdot)$ be $(\underline{t}_1, \bar{t}_1)$. Certainly $Y_1 \subseteq (\underline{t}_1, \bar{t}_1)$ (i.e., the values that ψ_1 can take must have meaning for the bearer of the risk, agent 1). Then letting $\lambda_1 \rightarrow 0$ and looking at (2.8), we see that $Z_1(\underline{y}, \lambda_1) \rightarrow \bar{t}_1 \forall \underline{y} \in Y$. Thus by choice of $\lambda_1 > 0$, we can make $Z_1(\underline{y}, \lambda_1) \geq y_1$, $\forall \underline{y} \in Y$. Similarly, by letting $\lambda_1 \rightarrow \infty$ we can make $Z_1(\underline{y}, \lambda_1) \leq y_1$, $\forall \underline{y} \in Y$. The last two observations, along with the continuity of $Z_1(\underline{y}, \cdot)$, guarantee the existence of a $\lambda_1 > 0$ so that (2.9) is satisfied.

4) Again heterogeneous (or subjective) probability density functions can be introduced without much difficulty. We only need to replace $\phi(\underline{y})$ by $\phi_1(\underline{y})$ in the market optimality conditions (2.8). Of course, we must further require that all $\phi_1(\underline{y})$ have a common domain as we did in 2.2.3/1.

2.4 The Existence of Equilibrium

Comparing (2.8) and (2.9) with (2.3) we immediately see that little is missing to make the contracts determined by the Market Optimality Conditions belong to the set of Pareto optimal treaties (defined by (2.3)). In fact what is missing from the market is the conservation or *clearing condition* $\sum_{i=1}^n Z_i = \sum_{i=1}^n y_i$. Recall that the reinsurance treaties determined by the Market Optimality Conditions depend on the given price function $P(\underline{y})$ of the market it is

natural therefore to ask: Does there exist a price function $P(\underline{y})$ such that the reinsurance treaty $Z(\underline{y})$ determined by the Market Optimality Conditions also satisfy the clearing condition?

This section is divided into two sub-sections. In the first (2.4.1) we formulate mathematically the problem discussed above and reformulate it as a problem of $(n - 1)$ equations in $(n - 1)$ unknowns. The approach only slightly differs from that by Borch (1968c). Our contribution comes with the second sub-section (2.4.2) where we prove the existence of a solution if certain assumptions are met.

2.4.1 The Formulation of the Problem

Problem I:

Find

$$P(\underline{y}) , Z_i(\underline{y}) , k_i \quad (i = 1, \dots, n)$$

so that

$$(2.10) \quad u'_i(Z_i(\underline{y})) = k_i \frac{P(\underline{y})}{\phi(\underline{y})} \quad (i = 1, \dots, n)$$

$$(2.11) \quad \int_{\underline{Y}} [Z_i(\underline{y}) - y_i] P(\underline{y}) d\underline{y} = 0$$

$$(2.12) \quad \sum_{i=1}^n Z_i(\underline{y}) = \sum_{i=1}^n y_i$$

$$(2.13) \quad k_i \geq 0 , \text{ at least one } k_i > 0 .$$

We already know (by Theorem 1, Sections 2.22 and 2.23/1,3) that the n equations (2.10) and (2.12) have a solution $P(\underline{y}, \underline{k})$,

$Z(\underline{y}, \underline{k})$ for all $k_i > 0$, $i = 1, \dots, n$ if assumption A.1 below holds.

Assumption A.1:

The $u_i'(\cdot)$ are strictly decreasing and continuous, with a range of $(0, \infty)$ and a domain $(\underline{t}_i, \bar{t}_i)$ where $(-\infty, +\infty)$ is allowed.

It follows that under A.1, Problem I will have a solution if Problem II has a solution.

Problem II:

Find

$$k_i > 0 \quad i = 1, \dots, n$$

such that

$$d_i(\underline{k}) = 0, \quad (i = 1, \dots, n)$$

where

$$(2.14) \quad d_i(\underline{k}) \equiv \int [Z_i(\underline{y}, \underline{k}) - y_i] P(\underline{y}, \underline{k}) d\underline{y}$$

where

$$(2.10)' \quad Z_i(\underline{y}, \underline{k}) = u_i^{-1} k_i \left(\frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})} \right)$$

because of (2.10) and A.1.

Call $d_i(\underline{k})$ the *reinsurance gain* of agent i . We will assume that $d_i(\underline{k})$ is well defined (i.e. the integral in (2.14) is finite) for all i for any choice of \underline{k} such that $0 < k_i < \infty$ $i = 1, \dots, n$.

One of the equations (2.14) is redundant since if $(n-1)$ are satisfied, so is the last by virtue of $\sum y_i = \sum Z_i$. On the other

hand, we only need to specify $(n - 1)$ of the k_i 's since, as we discussed in Section (2.2.3/4)

$$Z_i(\underline{y}, a\underline{k}) = Z_i(\underline{y}, \underline{k}) \quad \text{for any } a > 0 \quad i = 1, \dots, n$$

and

$$P(\underline{y}, a\underline{k}) = \frac{1}{a} P(\underline{y}, \underline{k}) .$$

This implies that Equations (2.14) hold for $a\underline{k}$ if they hold for \underline{k} , which means that we can only determine \underline{k} within a multiplicative constant. Thus there is a balance of $(n - 1)$ equations in $(n - 1)$ unknowns as is usual in market equilibrium models. We will therefore, with no loss of generality, restrict our search for \underline{k} in the set:

$$(2.15) \quad K \equiv \left\{ \underline{k} \mid 0 \leq k_i \leq 1, \sum_{i=1}^n k_i = 1 \right\}$$

which is convex and compact.

Borch (1968c) formulated Problem I in a slightly different form and transformed it into Problem II noting the balance of equations and unknowns. However he does not provide a proof for the existence of a \underline{k} such that Problem II has a solution. We attempt in the following sub-section to fill this gap by proving an existence theorem.

2.4.2 Existence of Solution

To prove the existence of a solution to Problem II we need the following Lemma and two more assumptions.

Lemma 1:

$P(\underline{y}, \underline{k})$ is continuous in \underline{k} for $0 < k_i \leq 1$; $i = 1, \dots, n$ if A.1 holds.

Proof:

By (2.10) and A.1

$$(2.14) \quad z_i(\underline{y}, \underline{k}) = u_i^{-1} k_i \left(\frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})} \right).$$

But by (2.12)

$$(2.16) \quad \int u_i^{-1} \left(k_i \frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})} \right) = \int y_i, \quad 0 < k_i \leq 1, \quad i = 1, \dots, n.$$

Since $u_i^{-1}(\cdot)$ are all continuous and strictly decreasing, a jump in $P(\underline{y}, \cdot)$ would not satisfy (2.15). ||

Continuity of $P(\underline{y}, \cdot)$ in the interior of \underline{k} is not enough. To use a fixed point theorem, as we plan to do below, we need some more assumptions on $P(\underline{y}, \underline{k})$. The assumptions we will make are sufficient but by no means necessary, as we will again discuss at the end of this section (Remarks/3).

First, we define the boundary ∂K of K as:

$$\partial K \equiv \{ \underline{k} \in K \mid \exists 1 \leq i \leq n, k_i = 0 \}.$$

Assumption A.2:

Let the random variable ψ_i take values on the interval Y_i and let $\underline{y}_i = \inf Y_i$, $\bar{y}_i = \sup Y_i$ then we assume that,

$$\underline{t}_i < \underline{y}_i < \bar{y}_i < \bar{t}_i \quad (i = 1, \dots, n).$$

Remark A.2.1

We demand here that agent 1 cannot initially possess an asset that can take values which have no meaning for him. (Recall $(\underline{t}_1, \bar{t}_1)$ is the domain of $u_1(\cdot)$ by A.1.) Further, A.2 requires that each y_1 vary over a finite interval even when $(\underline{t}_1, \bar{t}_1) = (-\infty, \infty)$, i.e. all the prereinsurance wealth of each agent can vary over a finite interval.

Assumption A.3:

For each $\underline{k}^* \in \partial K$

(a) $\frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})}$ has a limit $L(\underline{y}, \underline{k}^*)$.

(b) Let $Y_\infty(\underline{k}^*) \equiv \{\underline{y} \in Y \mid L(\underline{y}, \underline{k}^*) = +\infty\}$, $Y_0(\underline{k}^*) \equiv Y - Y_\infty(\underline{k}^*)$

where $\Pr(\underline{y} \in Y_\infty(\underline{k}^*)) > 0$ is allowed. Then we require that $\frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})}$ converges to $L(\underline{y}, \underline{k}^*)$ uniformly on each of the sets $Y_\infty(\underline{k}^*)$, $Y_0(\underline{k}^*)$.

Remark A.3.1:

The conservation condition (2.12)
$$\sum_{i=1}^n u_i^{-1} \left(k_i \frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})} \right) = \sum_{i=1}^n y_i$$

which must hold for all $\underline{y} \in Y$ and \underline{k} in the interior of K implies that zero cannot be a limit point (as $\underline{k} \rightarrow \underline{k}^*$) of

$\frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})}$ for any $\underline{y} \in Y$. For if the opposite were true then each

$u_i^{-1} \left(k_i \frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})} \right)$ would tend to its upper limit \bar{t}_i (since $u_i^{-1}(\cdot)$

strictly decreasing by assumption A.1). But by assumption A.2

$\bar{t}_i > \bar{y}_i \geq y_i$ for all i , and the conservation condition is violated.

Remark A.3.2:

The main point of assumption A.3 is that $\frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})}$ has a limit for each \underline{k}^0 on the boundary ∂K of K . However, it is possible that this limit is $+\infty$ for some \underline{y} . Let us try to explain what it means for $L(\underline{y}, \underline{k}^0)$ to be finite or $+\infty$. Consider the conditions for Pareto optimality (2.3)'. Let a k_1 tend to zero. If for some \underline{y} , $\frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})}$ remains finite in the limit, agent 1 achieves a contract Z_1 so that $u'_1(Z_1) = 0$. This implies that $Z_1 = \bar{t}_1$ for that \underline{y} . In other words he achieves his saturation point. The same happens to all agents 1 for which $k_1 \rightarrow 0$. (Call them over-demanding agents.) Further the rest of the agents, (the underdogs) for which $k_j^0 > 0$, will have $0 < u'_j(Z_j) < \infty$ which means that $\underline{t}_j < Z_j < \bar{t}_j$. But since $\sum_{i \in N} Z_i = \sum y_i$ we see that $\frac{P}{\phi}$ finite corresponds to a case when the underdogs can satisfy the over-demanding agents. The opposite happens when $L(\underline{y}, \underline{k}^0) = +\infty$ for some \underline{y} . Checking again (2.3)' we see that the underdogs' contracts Z_j tend to their lower limits \underline{t}_j (since $u'_j(Z_j) \rightarrow +\infty$) in an attempt to satisfy the overdemanding agents.

Return now to Problem II and define

$$(2.17) \quad \bar{d}_1(\underline{k}) \equiv \max(0, d_1(\underline{k}))$$

where $d_1(\underline{k})$ was defined in (2.11) and is the reinsurance gain of agent 1.

Also let

$$(2.18) \quad \bar{d}(\underline{k}) \equiv \sum_{i=1}^n \bar{d}_i(\underline{k})$$

$N \equiv \{1, \dots, n\}$ and $k_l \equiv \min_{i \in N} k_i$.

Certainly $l = l(\underline{k})$ and we will often denote it this way to remind us that it depends on \underline{k} . Further, we define

$$w_j(\underline{k}) \equiv \left| \bar{d}(\underline{k}) \left(k_j - k_{l(\underline{k})} \right) - \bar{d}_j(\underline{k}) \left(1 - n k_{l(\underline{k})} \right) \right|$$

(2.19)

$$w(\underline{k}) \equiv \sum_{j=1}^n w_j(\underline{k}).$$

$w(\underline{k})$ is well defined for \underline{k} in the interior of K . The following Lemma will tell us what happens on the boundary of K .

Lemma 2:

A.1, A.2, A.3 imply that for $\underline{k}^* \in \partial K$ $\lim_{\underline{k} \rightarrow \underline{k}^*} \bar{d}(\underline{k})$ and $\lim_{\underline{k} \rightarrow \underline{k}^*} w(\underline{k})$

either both exist and are strictly positive in which case for all $j \in N$ $\lim_{\underline{k} \rightarrow \underline{k}^*} \bar{d}_j(\underline{k})$ are also defined and $< \infty$ or they are both $+\infty$.

Proof: (See Appendix I)

Now $w(\underline{k})$ is continuous for \underline{k} in the interior of K . This is so because $k_{l(\underline{k})} = \min_{i \in N} k_i$ is continuous, $\bar{d}_j(\underline{k})$ is continuous since $P(y, \cdot)$ is continuous by Lemma 1, and therefore $\bar{d}(\underline{k})$ is continuous. It follows that $w_j(\underline{k})$ is continuous and thus so is $w(\underline{k})$. Therefore, we can extend the domain of definitions of $w(\underline{k})$, $\bar{d}(\underline{k})$ for points on the boundary of K by defining them to be equal to their limits. Thus, they will be continuous on K except where they are $+\infty$.

Theorem 2: (Existence Theorem)

If A.1, A.2, A.3 hold, Problem II has a solution.

Proof:

We use Brouwer's Fixed Point Theorem. This proof borrows from a similar proof by Arrow-Hahn (1971). (See Appendix I.)

Comments:

1) The existence theorem above is not restricted to reinsurance markets as we can think of ψ as the "state of nature". Then we can replace the initial fortune of agent 1, ψ_1 , by $f_1(\psi)$ and the market clearing condition changes from $\sum z_1 = \sum \psi_1$ to $\sum z_1 = \sum f_1$. The results continue to hold. This brings us into the more general area of securities markets for which the above theorem shows that if they are complete (i.e. an infinity of securities are available, that form a basis over the space of the "states of nature" as defined in Section 2.3.1), then equilibrium prices exist (or better, an equilibrium price function exists). It follows that if each agent who possesses positive or negative initial quantities of each security tries to maximize his expected utility subject to his budget constraint (which is the value of his initial quantities of securities), they reach a Pareto optimal result and the market is cleared.

2) The introduction of heterogeneous probabilities does not alter the result of the Theorem as long as Lemma 2 still holds. To prove Lemma 2 however A.3 must be replaced by a stronger version.

Namely, we need that A.3 must hold for each $\frac{P(y,k)}{\phi_1(y)}$; $i \in N$ and

further that for any pair $i, j \in N$ $\lim_{y \rightarrow y^*} \frac{\phi_i(y)}{\phi_j(y)}$ is finite for all y^* in the closure of Y .

This requirement says that there is no pair of agents one of which assigns zero probability while the other positive probability to the same event. This assumption is needed to insure that

the sets $\{Y_{\infty}^i(k^*)\}_{i=1}^n$ over which the $\lim_{k \rightarrow k^*} \frac{P(y, k)}{\phi_1(y)}$ is $+\infty$, co-

incide. The proof of Lemma 2 remains virtually the same.

3) Assumptions A.2, A.3 were made to determine the behavior of $\bar{d}(k)$ and $w(k)$ at the boundary of K . They are nothing more than sufficient conditions for the results of Lemma 2. Hence, if we could have the results of Lemma 2 by some other method, Theorem 2 would again follow.

In the examples below, A.1, A.2, A.3 hold if the y_i are bounded away from $+\infty$. However, we see that even if y_i can have a range of $(-\infty, \infty)$ (which violates A.2) a solution can exist if the appropriate expectations or integrals have meaning. This implies that there are more relaxed conditions than A.2 and A.3 that will be sufficient. But this is beyond the scope of this work.

2.5 Examples

Example 1: The quadratic case - revisited.

Consider n agents as in Section 2.1.4. The utility of agent i is

$$u_i(x) = -\frac{1}{2}x^2 + \beta_i x \quad (x \leq \beta_i) \quad (i = 1, \dots, n)$$

In Section 2.1.4, we concluded that

$$(2.20) \quad z_1(y, \underline{k}) = \frac{k_1}{\sum k_1} (y - \beta) + \beta_1$$

where

$$y \equiv \sum_{i=1}^n y_i, \quad \beta \equiv \sum_{i=1}^n \beta_i$$

also

$$(2.21) \quad \frac{P(y, \underline{k})}{\phi(y)} = \frac{\beta - y}{\sum k_1}.$$

We can assume $\sum_{i=1}^n k_i = 1$ without loss of generality, since the k_i can be found within a multiplicative constant. However, once an exogenously determined interest rate is given, a normalization is imposed on $P(y, \underline{k})$ and \underline{k}' should be chosen so that

$$\int P(y, \underline{k}') dy = \pi$$

where π is the value of a contract that pays to the holder \$1 under any circumstance.

Then by appropriate choice of $a > 0$,

$$\frac{1}{a} \int P(y, \underline{k}) dy = \pi$$

with $\sum k_i = 1$.

Thus the desired \underline{k}' are given by $\underline{k}' = a \underline{k}$, $\sum k_i = a$ since

$$\frac{1}{a} P(\underline{y}, \underline{k}) \equiv P(\underline{y}, a\underline{k}) , a > 0 .$$

Now we want to find $0 < k_i$'s so that the n equations:

$$(2.22) \quad d_i(\underline{k}) \equiv \int (Z_i - y_i) P(\underline{y}, \underline{k}) d\underline{y} = 0 \quad (i = 1, \dots, n)$$

are satisfied.

Substituting (2.20), (2.21) in (2.22) we obtain

$$\begin{aligned} d_i(\underline{k}) &\equiv \int [k_i(y - \beta) + \beta_i - y_i](\beta - y)\phi(\underline{y})d\underline{y} = \\ &- k_i E(\psi - \beta)^2 + \beta_i E(\beta - \psi) - \beta E\psi_i + E\psi\psi_i = 0 \\ (2.23) \quad &= -k_i E(\psi - \beta)^2 + E(\beta - \psi)(\beta_i - \psi_i) = 0 \\ \text{or } k_i &= \frac{E(\beta - \psi)(\beta_i - \psi_i)}{E(\psi - \beta)^2} \quad i = 1, \dots, n . \end{aligned}$$

Note that $k_i > 0$ because $E(\beta - \psi)(\beta_i - \psi_i) > 0$ if $\psi_i < \beta_i$ with probability one, for $i = 1, \dots, n$.

Also $\sum k_i = 1$ is satisfied.

Finally, substituting (2.23) back in (2.20):

$$(2.24) \quad Z_i(y, \underline{k}) = \frac{E(\beta - \psi)(\beta_i - \psi_i)}{E(\beta - \psi)^2} (y - \beta) + \beta_i$$

where k_i has the meaning of the proportion of the total asset that agent i takes.

Note that because of the form of $P(y, \underline{k})$, the only probabilistic quantities of interest are the second and first moments of ψ , ψ_i as well as $E\psi\psi_i$. These are enough to determine the quota of the total risk that agent i carries after reinsurance.

Example 2: The Exponential Case

Assume the utility function of agent i

$$u_i(x) = \frac{1}{\gamma_i} \left(1 - e^{-\gamma_i x} \right) \quad \gamma_i > 0, \quad -\infty < x < \infty \quad i = 1, \dots, n$$

then, substituting in the conditions for Pareto optimality (2.3), we have

$$(2.25) \quad e^{-\gamma_i z_i} = \frac{k_i}{k_j} e^{-\gamma_j z_j}$$

taking the log of both sides and summing over i

$$(2.26) \quad - \sum z_i = -y = \frac{\sum \log k_i}{\gamma_i} - \sum \frac{1}{\gamma_i} \gamma_j z_j - (\log k_j) \sum \frac{1}{\gamma_i}$$

$$\text{call } \sum \frac{1}{\gamma_i} \equiv \frac{1}{\gamma}$$

$$(2.27) \quad z_j = \frac{1/\gamma_j}{1/\gamma} \left(y + \frac{\sum \log k_i}{\gamma_i} \right) - \frac{\log k_j}{\gamma_j}.$$

z_j is linear in y (the pooled risky assets). This was expected as a partial case of a more general result found by Borch (1968b) who investigated the conditions on the utility functions of the agents so that linear sharing policies appear as a Pareto optimal allocation of risks. In this connection, he found that it was necessary that the agents' utilities belong to the *linear risk-tolerance* class of functions which will be defined later (Chapter 3).

Using (2.27) and the definition of $P(\underline{y}, \underline{k})$,

$$(2.3)' \quad \frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})} = \frac{u'_1(Z_1)}{k_j}$$

we find

$$(2.28) \quad \frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})} = e^{-y\gamma} \underbrace{\prod_{i=1}^n k_i^{-\gamma/\gamma_i}}_K$$

Note $P(\underline{y}, a\underline{k}) = \frac{1}{a} P(\underline{y}, \underline{k})$ as expected.

Also we see that the value of a contract $F(\underline{y})$ will be based on $E(e^{-\gamma\psi} F(\underline{y}))$, which reminds us of the Laplace transform.

Now let us determine k_1, \dots, k_n so that each agent's budget constraint is satisfied. We have the n equations:

$$d_j(\underline{k}) \equiv \int [Z_j - y_1] P(\underline{y}, \underline{k}) d\underline{y} = 0 \quad j = 1, \dots, n.$$

Substituting Z_j and P from above we attain

$$d_j(\underline{k}) = \frac{\gamma}{\gamma_j} E\psi e^{-\psi\gamma} + \frac{Ee^{-\psi\gamma}}{\gamma_j} \left(\log \prod_{i=1}^n k_i^{\gamma/\gamma_i} - \log k_j \right) - Ee^{-\psi\gamma} \psi_j = 0$$

and solving

$$(2.29) \quad \log k_j = \gamma \frac{E\psi e^{-\psi\gamma}}{Ee^{-\psi\gamma}} - \log K - \gamma_j \frac{Ee^{-\psi\gamma} \psi_j}{Ee^{-\psi\gamma}} \quad j = 1, \dots, n.$$

We can check that a solution is:

$$\log k_j = - \frac{\gamma_j E(e^{-\psi\gamma} \psi_j)}{Ee^{-\psi\gamma}}$$

and as required $k_j > 0$.

It should not surprise us that $\frac{P(y,k)}{\phi(y)}$ increases with y decreasing in both examples. In fact, this was expected by Section 2.2.3/5. But there is a physical meaning: the value of a certificate that pays \$1 when a great loss occurs ($\psi \rightarrow -\infty$) is very large.

Example 3:

This example is a slight modification of an example by Borch (1968c). Suppose there are only two agents in the market ($n = 2$) and $u_1(x) = x^{1/2}$, $u_2(x) = x^{3/4}$ and let ψ_1, ψ_2 their initial possession. Also $\phi_1(y_1, y_2)$ and $\phi_2(y_1, y_2)$ their subjective probability density functions for the random variables ψ_1, ψ_2 .

From (2.5) we have

$$k_2 z_1^{-1/2} \phi_1(y_1, y_2) = k_1 z_2^{-1/4} \phi_2(y_1, y_2).$$

Using $z_1 + z_2 = y \equiv y_1 + y_2$

$$(2.30) \quad z_1^2 - \left(\frac{k_2}{k_1}\right)^4 \left(\frac{\phi_1}{\phi_2}\right)^4 (y - z_1) = 0.$$

Calling $2h = \left(\frac{k_2}{k_1}\right)^4 F(y_1, y_2) = \left(\frac{\phi_1(y_1, y_2)}{\phi_2(y_1, y_2)}\right)^4$ we solve and obtain,

$$(2.31) \quad z_1(y_1, y_2) = \sqrt{h^2 F^2(y_1, y_2) + 2hF(y_1, y_2)y} - hF(y_1, y_2)$$

$$z_2(y_1, y_2) = y + hF(y_1, y_2) - \sqrt{h^2 F^2(y_1, y_2) + 2hF(y_1, y_2)y}$$

The price function is,

$$(2.32) \quad P(y_1, y_2) = \left(\sqrt{h^2 F^2 + 2hFy} - hF \right)^{1/2} \phi_1(y_1, y_2)^{\frac{1}{k_1}}$$

set $k_1 = 1$ without loss of generality and then the equilibrium equation becomes:

$$\int [Z_1 - y_1] P(y_1, y_2) dy_1, dy_2 = 0$$

or

$$(2.33) \quad 0 = \int \left[\sqrt{h^2 F^2(y_1, y_2) + 2hF(y_1, y_2)y} - hF(y_1, y_2) - y_1 \right] \left[\sqrt{h^2 F^2(y_1, y_2) + 2hF(y_1, y_2)y} - hF(y_1, y_2) \right]^{-1/2} \phi_1(y_1, y_2) dy_1, dy_2 .$$

We can check that the above equation of h has a unique real solution because the righthand side is strictly decreasing in k ; for $h \rightarrow 0$ it tends to $+\infty$ while for $h \rightarrow +\infty$ it tends to $-\infty$.

CHAPTER 3

THE DYNAMIC DECISION PROBLEM OF THE INSURANCE COMPANY

3.1 Introduction

A Reinsurance Market was defined in the last chapter. The principle idea is that a price function exists depending on the prereinsurance risks of the participating Insurance Companies, such that, when each Company tries to maximize its own expected utility, a Pareto optimal solution is reached. Thus, once an equilibrium price function is given, we can concentrate on the individual decision problem of a single company regarding it as a *price-taker* and not lose the cooperative nature of a mutual reinsurance treaty and the coupling due to the market mechanism. However, the results of this Chapter would be the same whether this Mutual Reinsurance Market exists or not; all that is required is that there exists a Reinsurer who calculates premiums for any contract using a price function. Details of the operation of such a reinsurer will be discussed in Section 4.7.1.

The purpose of this chapter will be to consider the dynamic decision problem of an Insurance Company that, in each period, collects known premiums for assuming risks and pays losses, which are random variables with known distributions. In addition, it pays dividends to its stock/policy-holders and reinsures in the Reinsurance Market or with a Reinsurer, as discussed above.

Our results differ considerably from previous approaches in the Insurance literature since the type of reinsurance is not arbitrarily chosen (for instance, stop loss, quota etc.) and then

the level calculated, but rather, the form of reinsurance will be directly related to the utility function of the company in question. To obtain simple closed form results we shall use the linear risk-tolerance class (LRT class) of utility functions.

3.2 Description of the Model

An Insurance Company (I.C.) is faced with an N-period decision problem. The decision variables are how much money to pay in dividends, and what type and level of reinsurance treaty to form given a Reinsurance Market is accessible each period. For convenience, we will number the periods backwards, such that the interval between points $[t, t-1]$ is the t^{th} interval, and the interval $[1, 0]$ is the first interval.

3.2.1 Definition of Quantities Entering the Formulation of the Problem.

Let n I.C.s participate in the Reinsurance Market in each period.[†] Call them IC_1, IC_2, \dots, IC_n .

ξ_t^1 : the losses occurring to IC_1 during period t if no mutual reinsurance takes place (a random variable whose value will be denoted by x_t^1).

$\xi_t \equiv \xi_t^1, \dots, \xi_t^n$: a vector random variable representing the vector of losses occurring to the I.C.s that participate in the Reinsurance Market. The value of ξ_t will be denoted by $x_t \in X_t$. We will assume that ξ_t is independent from period to period.

[†]The number of the participating companies can vary with t without affecting the results.

Let's now concentrate on IC_1 . Because of the definitions above, ξ_t^1 represents the losses occurring to IC_1 during period t if IC_1 does not participate in the Reinsurance Market. The value of the random variable ξ_t^1 will be, of course, x_t^1 . We will also need to define:

- R_t^1 : total reserves of IC_1 at start of period t just before dividends are paid. (Initial value for the problem is given.)
- C_t^1 : dividends paid to stockholders or premiums returned to policyholders.[†] (Decision variable.)
- P_t^1 : premiums received by IC_1 from its customers during period t . (Known in advance.)
- $\phi_t(\underline{x}_t)$: the joint probability density function of the n -dimensional vector of random variables $\{\xi_t^i\}_{i=1}^n$. (Known in advance.)^{††}
- $P_t(\underline{x}_t)$: the Reinsurance Market price function for period t . (Known in advance.)^{††}

We will assume that

$$0 < \frac{P_t(\underline{x}_t)}{\phi_t(\underline{x}_t)} < \infty \quad \underline{x}_t \in \text{closure of } X_t.$$

This is in accordance with the results of Chapter 2 and it

[†]Henceforth we will simply write dividends paid to s/p-holders to avoid this lengthy expression.

^{††}For the special class of utility functions which we will consider later, IC_1 will not need to know $P_t(\underline{x}_t)$ and $\phi_t^1(\underline{x}_t)$ explicitly. This will be discussed in Section 4.2 - Data Requirements.

insures that: (a) the price $P_t(\underline{x}_t)d\underline{x}_t$ of a certificate that pays \$1 when the event $E \equiv \{\xi_t \in (\underline{x}_t, \underline{x}_t + d\underline{x}_t]\}$ occurs and zero otherwise is not zero when $\phi_t(\underline{x}_t) > 0$ (i.e. when E has positive probability to occur) and it is zero when $\phi_t(\underline{x}_t) = 0$ (i.e. when E has zero probability of occurring); (b) $\int_{\underline{x}_t} P_t(\underline{x}_t)d\underline{x}_t$ is finite.

We will find useful to define

$$\pi_t \equiv \int P_t(\underline{x}_t)d\underline{x}_t.$$

To understand the meaning of π_t we can write the integral as follows:

$$\int P_t(\underline{x}_t)d\underline{x}_t = \int Z(\underline{x}_t)P_t(\underline{x}_t)d\underline{x}_t \quad \text{where } Z(\underline{x}_t) \equiv 1 \quad \forall \underline{x}_t$$

but then obviously the right hand side calculates the price of a contract that promises to pay \$1 at the end of the period irrespective of the realization of the random variable ξ_t . If π_t is to make sense in real life $P_t(\underline{x}_t)$ must be normalized so that $\pi_t < 1$. It follows that the interest rate in the Reinsurance Market is equal to $\frac{1 - \pi_t}{\pi_t}$ which we assume is the same in the whole economic structure that the Reinsurance Market is part of.

Returning now to I.C.₁ and assume for the moment that it did not enter the Reinsurance Market. Then its reserves process could be represented by

$$R_{t-1}^1 = \frac{R_t^1 - C_t^1}{\pi_t} + P_t^1 - \xi_t^1.$$

This is because at the start of period t I.C.₁ has R_t^1 as reserves; then it pays dividends C_t^1 to the s/p-holders leaving it with $R_t^1 - C_t^1$. This amount matures (grows) by the end of the period to $\frac{R_t^1 - C_t^1}{\pi_t}$ when the premiums P_t^1 are collected and the losses ξ_t^1 realize.[†] It then starts period $t-1$ with R_{t-1}^1 etc. Figure 1 depicts this process.

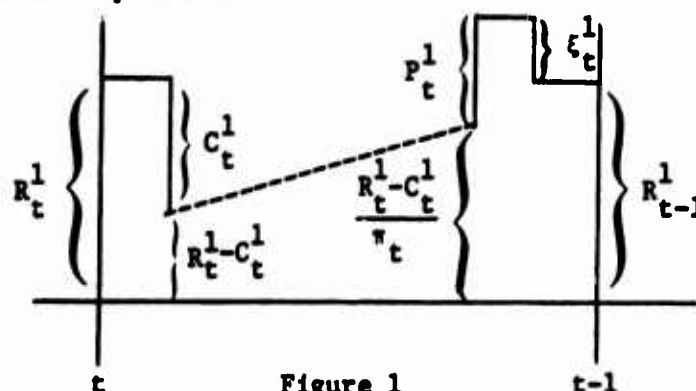


Figure 1
The prereinsurance reserves process of I.C.₁.

$$R_{t-1}^1 = \frac{R_t^1 - C_t^1}{\pi_t} + P_t^1 - \xi_t^1$$

3.2.2 The Objective of the Firm

We will assume that each I.C. is able to state its preferences over different dividends by use of a cardinal utility function

$$U(C_N, \dots, C_0), \quad (C_N = 0).$$

We will also assume that

$$U(C_N, \dots, C_0) = \sum_{k=0}^N \alpha^k u(C_{N-k})^{++} \quad 0 < \alpha < 1$$

[†] There are possible variations to this scenario to be discussed in Section 4.7.1.

⁺⁺ For justification of this form see T. C. Koopmans (1960), T. C. Koopmans, P. Diamond, R. Williamson (1964) and Meyer (1969).

where $u(\cdot)$ is an increasing, strictly concave, twice differentiable function. The quantity α is known as the *patience factor*.

Finally, faced with uncertain prospects, we assume that the I.C. will wish to maximize

$$EU(C_N, \dots, C_0)$$

the expected utility over the possible dividend streams.

U may be thought of as the utility of the management of the firm which inevitably should represent the s/p-holders whose interest is in the dividend stream. For example, we can think of U as the result of "profound" thinking of a "benevolent dictator" - the president of the firm, who formulates U as a function of the individual utilities of stockholders. For more details on how individual cardinal utility functions can be aggregated to form a group utility function the reader is referred to Keeney and Kirkwood (1973).

3.3 Formulation of the Decision Problem of the I.C. as a Dynamic Programming Problem

Convention:

To simplify notation in the rest of the Chapter we will now replace R_t^1, C_t^1, P_t^1 by R_t, C_t, P_t respectively and refer to I.C.₁ as simply the I.C.

The problem of the I.C. is to decide at the beginning of each period how much to pay as dividends and how to reinsure in the Re-insurance Market.

The problem can be formulated as a Dynamic Programming (D.P.) problem as follows:

Let

$f_t(R_t)$ = Optimal Value Function:

maximum expected utility starting period t with capital R_t to the end (period 0) .

The D.P. principle of optimality (Bellman (1957)) then states:

$$(3.1) \quad f_t(R_t) = \max_{C_t, R_{t-1}(\cdot)} [u(C_t) + \alpha E f_{t-1}(R_{t-1}(\xi_t))]$$

subject to the budget constraint:

$$(3.2) \quad \int_{\underline{X}_t} R_{t-1}(\underline{x}_t) P_t(\underline{x}_t) d\underline{x}_t = \int_{\underline{X}_t} \left(\frac{R_t - C_t}{\pi_t} + p_t - x_t^1 \right) P_t(\underline{x}_t) d\underline{x}_t$$

where, we recall,

$$(3.3) \quad \pi_t \equiv \int P_t(\underline{x}_t) d\underline{x}_t ; \pi_t < 1$$

and where $d\underline{x}_t \equiv dx_t^1 dx_t^2 \dots dx_t^n$.

At this point it is convenient to define the Reinsurance Market value of the losses of the I.C. as

$$(3.4) \quad \rho_t^1 \equiv \int x_t^1 P_t(\underline{x}_t) d\underline{x}_t$$

and again we will use the convention of dropping the superscript 1 and writing simply ρ_t .

The dividends C_t are not constrained to be nonnegative. This would be awkward in an investment consumption model of an individual. In the case of a firm though it is plausible. A more thorough discussion of this point as well as its relevance to the risk aversion

index will be found in Section 4.3.

The Boundary Condition (B.C.) for the D.P. relation is

$$(3.5) \quad f_0(R) = u(R)^+$$

and the answer to the N period problem will be $f_N(R_N)$ where R_N is the given starting capital.

3.4 Closed Form Solutions

It is evident that the D.P. approach presented is in general amenable only to numerical solutions.

However, it is the purpose of this section to show that for a class of utility functions known as the Linear Risk Tolerance Class (LRT), the problem has a closed form solution.

3.4.1 The Linear Risk-Tolerance (LRT) Class of Utility Functions

The quantity $-\frac{u''(x)}{u'(x)}$ has been called by Pratt (1964) the *risk-aversion*.^{††} The inverse $-\frac{u'(x)}{u''(x)}$ is called the *risk-tolerance*.

The LRT class of utility functions is defined by the real solutions to

$$(3.6) \quad \frac{u''(x)}{u'(x)} = \frac{f}{ax + b} \quad \begin{array}{l} f, a, b \text{ reals, and not both} \\ a = 0 \text{ and } b = 0. \end{array}$$

[†] This B.C. seems fine when R is positive. When R is negative, however, it requires that the I.C. or its shareholders pay the debt although the I.C. terminates operation.

Perhaps a B.C. of the form $f_0(R) = u(\max(0, R))$ would be more acceptable. However, apart from vastly complicating the problem, such a B.C. would induce the I.C. to borrow heavily and pay big dividends the last few periods before the end, with the prospect of immunity of debt upon termination. For this reason, we prefer (2.4) and its shortcomings.

^{††} For a discussion of the economic implications of this quantity see Arrow (1971).

Since we also demand $u''(x) < 0$ and $u'(x) > 0$, the solutions to (3.6) give rise to

I. $a \neq 0$. Then let $f = ca$

$$(3.7) \quad u'(x) = (ax + b)^c \quad ax + b > 0, \quad ac < 0$$

Class I can be split in two classes:

Ia. $c \neq -1$

$$(3.8) \quad u(x) = \frac{(ax + b)^{c+1}}{a(c+1)} \quad ; \quad ax + b > 0, \quad ac < 0$$

Ib. $c = -1$

$$(3.9) \quad u(x) = \frac{1}{a} \log(ax + b) \quad ; \quad a > 0, \quad x > -\frac{b}{a}$$

II. $a = 0$. Then let $\frac{f}{b} = -\gamma, \gamma > 0$

$$(3.10) \quad u'(x) = e^{-\gamma x}$$

$$(3.11) \quad u(x) = \frac{1}{\gamma}(1 - e^{-\gamma x}), \quad -\infty < x < \infty^+$$

For completeness we note that in both classes $u'(x)$ can vary on $(0, \infty)$ which is in accordance with the requirements of Chapter 2 on existence of market equilibrium.

Also, the utility functions we are considering are unbounded either from above or below or both, but this is not a shortcoming as long as we stay away from values which make them unbounded.⁺⁺

[†] Following the von Neumann Morgenstern axioms, the above utility functions are equivalent to their positive linear transformations: $u \sim au + b, a > 0$.

⁺⁺ For a discussion of this point, see for example Hakansson's thesis (1966), p. 23.

The LRT class was already mentioned in Section 2.5 (Example 2) as being the only class that produces linear sharing policies in Pareto optimal reinsurance treaties (Borch (1968b)). It is also well known in portfolio selection theories for being closely related to the *separation* property (Hakansson (1969b), Cass and Stiglitz (1970)).

We will sometimes find it useful to distinguish class Ia in three subclasses:

$$Ia_1: a > 0, c < -1$$

$$Ia_2: a > 0, -1 < c < 0$$

$$Ia_3: a < 0, c > 0$$

Hakansson has used the same class of functions in his studies on the sequential assumption investment problem of the individual, (1966), (1969a), (1969b), (1970a), (1970b). As a consequence the models and results of this thesis resemble those of Hakansson. In fact under certain conditions our problem can take the form of Hakansson's problem. The major difference lies with our assumption that the market is complete. Thus instead of proving the existence of an optimal mix of securities (or the post-reinsurance unit of risky asset) we are able to find it explicitly using the market optimality conditions (2.8) and (2.9). The similarity of our model to that of Hakansson becomes obvious when we expand $R(\underline{x})$ and $P(\underline{x})$ as a linear combination of a complete orthonormal set of functions. (For more details on how to achieve the transformation as well as the conditions required see Appendix IV.)

3.4.2 Model Ia

For this Model only we will proceed with detail through the derivation of the results which appear at the end of this section starting with (3.4.2). For this as well as the other Models the results will be discussed in detail in Chapter 4.

3.4.2.1 One Period Solution

We look at the last period (period 1) of the D.P. relation (3.1), (3.2) and using the Boundary Condition, we have:

$$(3.12) \quad f_1(R_1) = \max_{C_1, R_0(\cdot)} [u(C_1) + \alpha E u(R_0(\xi_1))]$$

such that,

$$(3.13) \quad \int_{x_1} R_0(x_1) P_1(x_1) dx_1 = \int_{x_1} \left(\frac{R_1 - C_1}{\pi_1} + p_1 - x_1^1 \right) P_1(x_1) dx_1$$

Fix C_1 .

Thus the maximization problem of the second term in (3.12) is equivalent to maximization in the market with price function $P_1(x_1)$. We know therefore from Chapter 2 (2.8) that the optimal reinsurance treaty $R_0(x_1)$ must satisfy:

$$(3.14) \quad u'(R_0(x_1)) \phi_1(x_1) = \lambda P_1(x_1)$$

and (3.13).

Now for model Ia we have

$$u'(x) = (ax + b)^c \quad c \neq -1, \quad ac < 0, \quad ax + b > 0$$

which has a range of $(0, \infty)$ and is strictly decreasing. Thus

(3.14) always has a unique solution as long as $\lambda > 0$.

Then solving (3.14) for Model Ia, we find that the optimal reinsurance for one period (fixed C_1) should be so that the terminal reserve is:

$$(3.15) \quad R_0(\underline{x}_1) = \frac{\lambda^{1/c}}{a} \left(\frac{P_1(\underline{x}_1)}{\phi_1(\underline{x}_1)} \right)^{1/c} - \frac{b}{a}$$

Substituting (3.14) in (3.12) we find that the market multiplier

$$(3.16) \quad \lambda^{1/c} = \frac{a(R_1 - C_1 + p_1\pi_1 + \frac{b}{a}\pi_1 - \rho_1)}{m_1}$$

where ρ_1 was defined in (3.4) as the value in the Reinsurance Market of the prereinsurance claims portfolio of the I.C. and m_1 is defined by

$$(3.17) \quad m_t \equiv \int \left(\frac{P_t(\underline{x}_t)}{\phi_t(\underline{x}_t)} \right)^{1/c} P_t(\underline{x}_t) d\underline{x}_t ; \quad m_t > 0 \quad \text{since } P_t > 0, \phi_t > 0.$$

We will often refer to $\left(\frac{P_t(\xi_t)}{\phi_t(\xi_t)} \right)^{1/c}$ (a random variable) as a

unit of the postreinsurance risky asset of the I.C. when we talk about model Ia. The name is justified by looking at (3.15) and for the general case at (3.34). In this sense m_t defined in (3.17) above can be thought of as the *cost of a unit of postreinsurance risky asset* for period t .

Substituting now (3.15) in the maximand of (3.12):

$$(3.18) \quad f_1(R_1) = \max_{C_1} \left\{ u(C_1) + \alpha E u \left[\frac{\lambda^{1/c} \left(\frac{P_1}{\phi_1} \right)^{1/c}}{a} - \frac{b}{a} \right] \right\}$$

and since $u(\cdot)$ is of type Ia, the D.P. relation takes the form:

$$(3.19) \quad f_1(R_1) = \max_{C_1} \left\{ u(C_1) + \frac{\alpha \lambda^{c+1/c}}{a(c+1)} E \left[\left(\frac{P_1(x_1)}{\phi_1(x_1)} \right)^{c+1/c} \right] \right\}$$

but we note that for any t ,

$$E \left[\left(\frac{P_t(\xi_t)}{\phi_t(\xi_t)} \right)^{c+1/c} \right] = \int \frac{P}{\phi} \left(\frac{P}{\phi} \right)^{1/c} d\underline{x}_t = \int \left(\frac{P}{\phi} \right)^{1/c} P d\underline{x}_t \equiv m_t$$

because of (3.17). Thus, after we substitute for $\lambda^{1/c}$ from (3.15), the one period D.P. problem reduces to:

$$(3.20) \quad f_1(R_1) = \max_{C_1} \left\{ u(C_1) + \frac{\alpha m_1}{a(c+1)} \left[\frac{a}{m_1} (R_1 - C_1) + \frac{p_1 \pi_1 a + b \pi_1 - a \rho_1}{m_1} \right]^{c+1} \right\}$$

The second term in the maximand is concave in C_1 as long as

$$(3.21) \quad a(R_1 - C_1 + p_1 \pi_1 - \rho_1) + b \pi_1 > 0$$

which is implied by $\lambda > 0$ which is required for (3.14) to have a solution.

Thus the maximand of (3.20) is concave in C_1 as long as:

(i) $aC_1 + b > 0$, which guarantees that the first term has meaning and is concave; and (ii) (3.21) holds which guarantees the second term is concave. Assume that these conditions hold. Differentiating the maximand of (3.20) wrt, C_1 , and setting it equal to zero, we

find that the unique optimal dividend C_1^* is given by:

$$(3.22) \quad aC_1^* + b = \alpha^{1/c} \left[\frac{a}{m_1} (R_1 - C_1) + \frac{p_1 \pi_1 a + b \pi_1 - a \rho_1}{m_1} \right]$$

or

$$(3.23) \quad C_1^* = \frac{\frac{\alpha^{1/c}}{m_1}}{1 + \frac{\alpha^{1/c}}{m_1}} R_1 + \frac{\frac{\alpha^{1/c}}{m_1}}{1 + \frac{\alpha^{1/c}}{m_1}} \left[p_1 \pi_1 + \frac{b}{a} \pi_1 - \rho_1 - \frac{b m_1}{a \alpha^{1/c}} \right]$$

which is linear in R_1 !

But now because of (3.22), conditions (i) and (ii) above are equivalent and thus we only demand that the optimal dividend C_1^* satisfy

$$(3.24) \quad aC_1^* + b > 0.$$

Finally, substituting C_1^* in (3.20) the maximum expected utility of the one period problem when an optimal policy is followed with respect to reinsurance and dividend payment is given by:

$$\begin{aligned} f_1(R_1) &= u(C_1^*) + \frac{\alpha m_1}{a(c+1)} \underbrace{\left[\frac{a}{m_1} (R_1 - C_1^*) + \frac{p_1 \pi_1 a + b \pi_1 - a \rho_1}{m_1} \right]}_{\substack{aC_1^* + b \\ \alpha^{1/c} \text{ by (3.22)}}}^{c+1} \\ &= u(C_1^*) + \frac{\alpha m_1}{\alpha^{c+1/c}} u(C_1^*) \\ &= \left(1 + \frac{m_1}{\alpha^{1/c}} \right) u(C_1^*). \end{aligned}$$

Thus inspired by (3.23) we let $C_1^* = A_1 R_1 + B_1$ and then

$$(3.25) \quad f_1(R_1) = D_1 u(A_1 R_1 + B_1)$$

where

$$\left. \begin{aligned} D_1 &= 1 + \frac{m_1}{\alpha^{1/c}} \\ A_1 &= \frac{\alpha^{1/c/m_1}}{1 + \alpha^{1/c/m_1}} \end{aligned} \right\} \quad D_1 A_1 = 1, \quad 0 \leq A_1 \leq 1, \quad D_1 \geq 1$$

$$B_1 = A_1 \left[\rho_1 \pi_1 - \rho_1 + \frac{b}{a} \left(\pi_1 - \frac{m_1}{\alpha^{1/c}} \right) \right]$$

along with the condition:

$$(3.24) \quad a(A_1 R_1 + B_1) + b > 0$$

which simply demands that R_1 must be such that $u(A_1 R_1 + B_1)$ has meaning. This is so when we recall that $u(x)$ was defined for $ax + b > 0$.

To summarize, C_1^* is given by (3.23), the reinsurance treaty is defined by (3.15), (3.16), and the maximum expected utility, which is realized when these policies are followed is given by (3.25), (3.26).

3.3.2.2 Extension to N-periods

We now proceed inductively. The arguments are similar.

Suppose we solved the t period problem and we found

$$\begin{aligned}
 f_t(R_t) &= D_t u(A_t R_t + B_t) \\
 (3.27) \quad D_t A_t &= 1, D_t \geq 0, A_t \geq 0
 \end{aligned}$$

and the only condition needed was

$$(3.28) \quad a(A_t R_t + B_t) + b > 0.$$

That is, no new condition arise in steps $t-1, t-2, \dots, 1$.

Then we will show that

$$\begin{aligned}
 f_{t+1}(R_{t+1}) &= D_{t+1} u(A_{t+1} R_{t+1} + B_{t+1}), \\
 (3.29) \quad D_{t+1} A_{t+1} &= 1, D_{t+1} \geq 0, A_{t+1} \geq 0
 \end{aligned}$$

and the only condition needed is

$$(3.30) \quad a(A_{t+1} R_{t+1} + B_{t+1}) + b > 0.$$

The latter will be established when we show that (3.30) and optimization in step $t+1$ imply (3.28).

We write again (3.1), (3.2) for period $t+1$:

$$(3.1) \quad f_{t+1}(R_{t+1}) = \max_{C_{t+1}, R_t(\cdot)} \{u(C_{t+1}) + \alpha E f_t(R_t(\xi_{t+1}))\}$$

s.t.

$$\begin{aligned}
 &\int R_t(x_{t+1}) P_{t+1}(x_{t+1}) dx_{t+1} = \\
 (3.2) \quad &\int \left[\frac{R_{t+1} - C_{t+1}}{\pi_{t+1}} + p_{t+1} - x_{t+1}^1 \right] P_{t+1}(x_{t+1}) dx_{t+1}
 \end{aligned}$$

Fix C_{t+1} . Use (3.27) and maximize in the market with price function $P_{t+1}(\underline{x}_{t+1})$ to find $R_t(\underline{x}_{t+1})$:

$$(3.31) \quad \frac{d}{dR_t} f_t(R_t) \phi_{t+1}(\underline{x}_{t+1}) = \lambda P_{t+1}(\underline{x}_{t+1})$$

or

$$(3.32) \quad D_t A_t u'(A_t R_t + B_t) = \lambda \frac{P_{t+1}(\underline{x}_{t+1})}{\phi_{t+1}(\underline{x}_{t+1})}$$

or

$$(3.33) \quad A_t R_t + B_t = \frac{1}{a} \lambda^{1/c} \left(\frac{P_{t+1}}{\phi_{t+1}} \right)^{1/c} - \frac{b}{a}$$

or

$$(3.34) \quad R_t(\underline{x}_{t+1}) = \frac{\lambda^{1/c} \left(\frac{P_{t+1}(\underline{x}_{t+1})}{\phi_{t+1}(\underline{x}_{t+1})} \right)^{1/c}}{a A_t} - \frac{b}{a A_t} - \frac{B_t}{A_t}$$

Substituting (3.34) in the budget constraint (3.2), we solve for $\lambda^{1/c}$ (of $(t+1)^{st}$ period):

$$(3.35) \quad \lambda^{1/c} = (R_{t+1}) \frac{a A_t}{w_{t+1}} + (p_{t+1} \pi_{t+1} - \rho_{t+1}) \frac{a A_t}{m_{t+1}} + \frac{b \pi_{t+1}}{m_{t+1}} + \frac{B_t \pi_{t+1}^a}{m_{t+1}}$$

For convenience of notation and because there is no danger of confusion, we replace $p_{t+1}, \rho_{t+1}, m_{t+1}, \pi_{t+1}, P_{t+1}(\underline{x}_{t+1}), \phi_{t+1}(\underline{x}_{t+1})$, by p, ρ, m, π, P, ϕ respectively. Now the condition $\lambda^{1/c} > 0$ (so that

3.32 has meaning) becomes by (3.35):

$$(3.36) \quad (R_{t+1} - C_{t+1} + p\pi - \rho)aA_t + b\pi + B_t\pi a > 0.$$

Now substituting (3.35) and (3.33) in (3.1) we get

$$(3.37) \quad f_{t+1}(R_{t+1}) = \max_{C_{t+1}} \left\{ u(C_{t+1}) + \frac{\alpha D_t m}{a(c+1)} \left[\frac{aA_t}{m} (R_{t+1} - C_{t+1} + p\pi - \rho) + \frac{b\pi}{m} + B_t \frac{\pi a}{m} \right]^{c+1} \right\}.$$

The first term in the maximand above is strictly concave as long as

$$(3.38) \quad aC_{t+1} + b > 0$$

and the second term is also strictly concave (shown by twice differentiating) as long as (3.36) holds.

Thus differentiating w.r.t. C_{t+1} and setting equal to zero, we are guaranteed of a unique maximum:

$$(3.39) \quad aC_{t+1}^* + b = (\alpha D_t A_t)^{1/c} \left[\frac{aA_t}{m} (R_{t+1} - C_{t+1}^* + p\pi - \rho) + \frac{b}{m} + \frac{B_t \pi a}{m} \right]$$

or using $D_t A_t = 1$ by assumption

$$(3.40) \quad C_{t+1}^* = \frac{\frac{A_t \alpha^{1/c}}{m}}{1 + \frac{A_t \alpha^{1/c}}{m}} R_{t+1} + \frac{\frac{A_t \alpha^{1/c}}{m}}{1 + \frac{A_t \alpha^{1/c}}{m}} \left[p\pi - \rho + \frac{b\pi}{aA_t} + \frac{B_t \pi}{A_t} - \frac{bm}{aA_t \alpha^{1/c}} \right]$$

and thus the optimal dividend C_{t+1}^* is again linear in R_{t+1} , the reserves at the start of the period. Because of (3.39), conditions (3.38) and (3.36) are equivalent. Thus, the only condition needed

is that the optimal dividend must satisfy:

$$(3.38) \quad aC_{t+1}^* + b > 0.$$

Now let us summarize the conditions:

$$(3.41) \quad (3.38) \Leftrightarrow (3.36) \Leftrightarrow \lambda^{1/c} > 0 \Leftrightarrow a(A_t R_t + B_t) + b > 0$$

\uparrow
 because of
choice of
 C_{t+1}^*
(3.39)

\uparrow
 because of
budget
constraint
(3.35)

\uparrow
 because of
choice of
 R_t
(3.33)

Finally, because of (3.39), (3.37) becomes:

$$f_{t+1}(R_{t+1}) = u(C_{t+1}^*) + \frac{\alpha D_t^m}{a(c+1)} \left[\frac{a A_t}{m} (R_{t+1} - C_{t+1}^* + p\pi - \rho) + \frac{b\pi}{m} + \frac{B_t \pi a}{m} \right]^{c+1}$$

$\underbrace{\hspace{15em}}$
 $\frac{aC_{t+1}^* + b}{a^{1/c}} \text{ by (3.39)}$

$$f_{t+1}(R_{t+1}) = \left[1 + \frac{\alpha D_t^m}{a^{c+1/c}} \right] u(C_{t+1}^*)$$

or

$$(3.42) \quad f_{t+1}(R_{t+1}) = D_{t+1} u(A_{t+1} R_{t+1} + B_{t+1})$$

which is again of the promised form, with,

$$(3.43a) \quad D_{t+1} = 1 + \frac{D_t^m}{a^{1/c}}$$

$$(3.43b) \quad A_{t+1} = \frac{\frac{A_t a^{1/c}}{m_{t+1}}}{1 + \frac{A_t a^{1/c}}{m_{t+1}}}$$

$$\left. \begin{array}{l} (3.43a) \\ (3.43b) \end{array} \right\} D_{t+1} A_{t+1} = 1, \quad 0 \leq A_{t+1} \leq 1, \quad D_{t+1} \geq 1$$

$$(3.43c) \quad B_{t+1} = A_{t+1} \left[P_{t+1} \pi_{t+1} - \rho_{t+1} + \frac{b \pi_{t+1}}{a A_t} + \frac{B_t \pi_{t+1}}{A_t} - \frac{b m_{t+1}}{a A_t^{1/c}} \right]$$

Note that we are now in position to calculate all the A_t, D_t, B_t recursively, starting with $A_0 = 1$, $D_0 = 1$, $B_0 = 0$.

The only condition needed for maximization in step $t + 1$ is:

$$(3.38) \quad a C_{t+1}^* + b > 0$$

where by (3.40)

$$(3.40)' \quad C_{t+1}^* = A_{t+1} R_{t+1} + B_{t+1}.$$

Then (3.38) is equivalent to:

$$(3.30) \quad a(A_{t+1} R_{t+1} + B_{t+1}) + b > 0.$$

But looking back at the summary of conditions in (3.41), we see that

$$(3.30) \iff (3.38) \iff a(A_t R_t + B_t) + b > 0.$$

Thus the condition for the t step (3.28) is implied by (3.30) (the condition for the $t + 1$ step) and no other condition is needed. The optimal reinsurance treaty is now given by:

$$(3.34) \quad R_t(\xi) = \frac{\lambda^{1/c}}{a A_t} \left(\frac{P_{t+1}(\xi)}{P_{t+1}(\xi)} \right)^{1/c} - \frac{b}{a A_t} - \frac{B_t}{A_t}$$

where because of (3.40)', $\lambda^{1/c}$ is given by

$$(3.44) \quad \frac{\lambda^{1/c}}{a A_t} = \frac{1}{m_{t+1}} \left[(1 - A_{t+1}) \left(R_{t+1} - P_{t+1} \rho_{t+1} - \tau_{t+1} + \frac{b \pi_{t+1}}{a A_t} + \frac{B_t}{A_t} \pi_{t+1} \right) + \frac{A_{t+1}}{A_t} \frac{b}{a} \frac{m_{t+1}}{a^{1/c}} \right].$$

We will discuss these results in Chapter 4.

3.4.3 Model Ib

Because of the similarity of the method to that used for Model Ia we quickly sketch the results for the other Models.

In Model Ib

$$u(x) = \frac{1}{a} \log(ax + b) ; a > 0 , ax + b > 0$$

$$u'(x) = (ax + b)^{-1} .$$

Following similar arguments as in Model Ia we find that

$$(3.45) \quad f_{t+1}(R_{t+1}) = D_{t+1} u(A_{t+1} R_{t+1} + B_{t+1}) + E_{t+1}$$

and

$$(3.46) \quad C_{t+1}^* = A_{t+1} R_{t+1} + B_{t+1}$$

is again linear in R_{t+1} , where

$$(3.47a) \quad D_{t+1} = 1 + \alpha D_t \quad \left\{ \begin{array}{l} A_{t+1} D_{t+1} = 1 , \quad 0 \leq A_{t+1} \leq 1 , \quad D_{t+1} \geq 1 \\ (3.47b) \quad A_{t+1} = \frac{A_t / \alpha}{1 + A_t / \alpha} \end{array} \right.$$

$$(3.47c) \quad B_{t+1} = A_{t+1} \left[p_{t+1} \pi_{t+1} - \rho_{t+1} + \frac{B_t}{A_t} \pi_{t+1} + \frac{b \pi_{t+1}}{a A_t} - \alpha \frac{b}{a A_t} \right]$$

$$(3.47d) \quad E_{t+1} = \frac{\alpha}{a} D_t [\log + q_{t+1}] + \alpha E_t$$

with boundary conditions,

$$(3.48) \quad A_0 = 1, D_0 = 1, B_0 = 0, E_0 = 0$$

As before ρ_{t+1} is given by (3.4) and has exactly the same meaning. However here the quantity

$$(3.49) \quad q_{t+1} \equiv E \left[\log \frac{\phi_{t+1}(\xi_{t+1})}{P_{t+1}(\xi_{t+1})} \right]$$

enters the Model but only in the recursive constant E_{t+1} and in no way affects the optimal policies. For this reason, the I.C. does not need to calculate this quantity (see Section 4.2). The only condition we need is

$$(3.50) \quad a(A_{t+1}R_{t+1} + B_{t+1}) + b > 0.$$

Finally, the optimal reinsurance treaty is:

$$(3.51) \quad R_t(x_{t+1}) = \frac{1}{aA_t \lambda} \frac{\phi_{t+1}(x_{t+1})}{P_{t+1}(x_{t+1})} - \frac{b}{aA_t} - \frac{B_t}{A_t}.$$

Similarly to model Ia we will refer to $\frac{\phi_{t+1}(\xi_{t+1})}{P_{t+1}(\xi_{t+1})}$ as a unit of postreinsurance risky asset for period $t+1$. Then substituting (3.51) in the budget constraint we obtain,

$$(3.52) \quad \frac{1}{a\lambda A_t} = R_{t+1} - C_{t+1} + P_{t+1}\pi_{t+1} - \rho_{t+1} + \frac{b}{aA_t}\pi_{t+1} + \frac{B_t}{A_t}\pi_{t+1}$$

and because of (3.46) and the formula for B_{t+1} in (3.47c),

$$(3.53) \quad \frac{1}{a\lambda A_t} = (1 - A_{t+1}) \left(R_{t+1} + P_{t+1}\pi_{t+1} - \rho_{t+1} + \frac{b}{aA_t}\pi_{t+1} + \frac{B_t}{A_t}\pi_{t+1} \right) + a \frac{A_{t+1}}{A_t} \frac{b}{a}.$$

Note that all the formulas in Model Ib except for (3.45), (3.47d), (3.48) can be derived from the respective formulas of Model Ia by setting $c = -1$ and $m_{t+1} = 1$. Indeed m_{t+1} the cost of a unit of risky asset is 1 for Model Ib because

$$\int \frac{\phi_{t+1}(\underline{x}_{t+1})}{P_{t+1}(\underline{x}_{t+1})} P_{t+1}(\underline{x}_{t+1}) d\underline{x}_{t+1} = 1$$

A sketch of the induction step of the proof follows:

Given that the formulae hold for t , we start from the D.P. relation (3.1), (3.2). Fix C_{t+1} and optimize $f_t(R_t)$ over $R_t(\cdot)$ in the market $P_{t+1}(\underline{x}_{t+1})$:

$$\frac{d}{dR_t} f'_t(R_t) \phi_{t+1}(\underline{x}_{t+1}) = \lambda P_{t+1}(\underline{x}_{t+1})$$

from which it follows that

$$(3.51) \quad R_t(\underline{x}_{t+1}) = \frac{1}{a\lambda A_t} \frac{\phi_{t+1}(\underline{x}_{t+1})}{P_{t+1}(\underline{x}_{t+1})} - \frac{b}{aA_t} - \frac{B_t}{A_t}$$

and because of the budget constraint (3.2),

$$\frac{1}{a\lambda A_t} = R_{t+1} - C_{t+1} + P_{t+1}\pi_{t+1} - \rho_{t+1} + \frac{b}{aA_t}\pi_{t+1} + \frac{B_t}{A_t}\pi_{t+1}.$$

Then substituting $R_t(\underline{x}_{t+1})$ in the D.P. relation, we see that the maximand (now over C_{t+1}) is concave over C_{t+1} .

Optimization w.r.t. C_{t+1} gives us

$$C_{t+1}^* = A_{t+1}R_{t+1} + B_{t+1}$$

with A_{t+1} , B_{t+1} as defined above and the only condition required for the $t+1$ period problem is

$$a(A_{t+1}R_{t+1} + B_{t+1}) + b > 0.$$

The solution is also unique both in C_{t+1} , $R_t(x)$ since the functions that are being maximized in each case are strictly concave by choice of $u(x)$.

3.4.4 Model II

Here

$$u(x) = \frac{1}{\gamma} (1 - e^{-\gamma x})$$

$$u'(x) = e^{-\gamma x} \quad -\infty < x < \infty, \gamma > 0.$$

Along similar arguments as before we can prove:

$$(3.54) \quad f_{t+1}(R_{t+1}) = D_{t+1} u(A_{t+1}R_{t+1} + B_{t+1}) + E_{t+1}$$

and

$$(3.55) \quad C_{t+1}^* = A_{t+1}R_{t+1} + B_{t+1}$$

which is once more linear in R_{t+1} , with,

$$(3.56a) \quad \left. \begin{aligned} D_{t+1} &= 1 + \pi_{t+1} D_t \\ A_{t+1} &= \frac{A_t / \pi_{t+1}}{1 + A_t / \pi_{t+1}} \end{aligned} \right\} \begin{aligned} A_{t+1} D_{t+1} &= 1, \quad 0 \leq A_{t+1} \leq 1, \quad D_{t+1} \geq 1 \end{aligned}$$

$$(3.56c) \quad B_{t+1} = A_{t+1} \left[p_{t+1} \pi_{t+1} - \rho_{t+1} + \frac{B_t}{A_t} \pi_{t+1} + \frac{w_{t+1}}{\gamma A_t} - \frac{\pi_{t+1}}{\gamma A_t} \log \alpha \right]$$

$$(3.56d) \quad E_{t+1} = \frac{D_t}{\gamma} [\alpha - \pi_{t+1}] + \alpha E_t,$$

which are calculated recursively, starting with

$$(3.57) \quad D_0 = 1, A_0 = 1, B_0 = 0, E_0 = 0$$

As before, ρ_{t+1} , the value of the claims portfolio of the I.C. in the Reinsurance Market is given by (3.4)

$$\text{Motivated by (3.59) we will refer to } \log \frac{P_{t+1}(\xi_{t+1})}{\phi_{t+1}(\xi_{t+1})} \text{ as a}$$

unit of *post-reinsurance risky asset*, and define:

$$(3.58) \quad w_{t+1} \equiv \int \left(\log \frac{P_{t+1}(x_{t+1})}{\phi_{t+1}(x_{t+1})} \right) P_{t+1}(x_{t+1}) dx_{t+1}$$

as the *cost of a unit of risky asset* for period $t+1$ for Model II.

In this Model, no secondary condition is imposed for the problem to have this solution; and finally, the optimal reinsurance treaty is:

$$(3.59) \quad R_t(\xi_{t+1}) = -\frac{1}{\gamma A_t} \log \left(\lambda \frac{P_{t+1}(\xi_{t+1})}{\phi_{t+1}(\xi_{t+1})} \right) - \frac{B_t}{A_t}$$

where

$$(3.60) \quad -\frac{\log \lambda}{\gamma A_t} = \frac{1}{\pi_{t+1}} \left[R_{t+1} - C_{t+1}^* + p_{t+1} \pi_{t+1} - \rho_{t+1} \right] + \frac{B_t}{A_t} + \frac{w_{t+1}}{\gamma A_t \pi_{t+1}}$$

or, after some manipulations using (3.55) and (3.56):

$$(3.61) \quad R_t(\xi_{t+1}) = \frac{1}{\pi_{t+1}} (1 - A_{t+1}) \left(R_{t+1} + p_{t+1} \pi_{t+1} - \rho_{t+1} + \frac{w_{t+1}}{\gamma A_t} \right) - A_{t+1} \frac{B_t}{A_t} + \frac{A_{t+1}}{A_t} \log \alpha - \frac{1}{\gamma A_t} \log \frac{P_{t+1}(\xi_{t+1})}{\phi_{t+1}(\xi_{t+1})}$$

The proof for the above is rather similar to the ones for the previous models and only an outline of the induction step (from t to $t + 1$) will be presented here.

Starting with the D.P. relation (3.1), (3.2), we fix C_{t+1} and maximize $f_t(R_t)$ over $R_t(\cdot)$ in the market with $P_{t+1}(\underline{x}_{t+1})$ thus obtaining (3.59).

Then using the budget constraint (3.2) we get (3.60). We substitute (3.59), (3.60) in (3.2) and observing that now the maximand (over C_{t+1}) is strictly concave in C_{t+1} we differentiate and find

$$C_{t+1}^* = A_{t+1}R_{t+1} + B_{t+1}$$

with A_{t+1} , B_{t+1} defined as in (3.56).

Substituting back in the maximand of (3.1) we get (3.54).

The solution is also unique because of the strict concavity of $u(x)$.

The interpretation and comparison of these results for the various Models will continue in the next Chapter.

CHAPTER 4

INTERPRETATION OF RESULTS AND PROPERTIES OF OPTIMAL POLICIES; GENERALIZATIONS

4.1 Introduction

This Chapter consists of two main parts. The first part (Sections 4.2 to 4.6) interprets the results found in Chapter 3. Specifically, in Section 4.2 we exemplify the data requirements of the problem. In Sections 4.3 and 4.4 we explain the meaning and restrictions of the optimal dividend and reinsurance policies. Some interesting inequalities are developed in Section 4.5. Finally, in Section 4.6, extension is made to the infinite horizon stationary case.

The second part (Section 4.7) first reviews some simple generalizations already mentioned sporadically and then introduces two others. The first deals with the decision to liquidate the I.C. and the second introduces an additional decision variable to the problem - expenditures for promotion of sales.

4.2 Data Requirements[†]

We mentioned in Section 3.2.1 that the I.C. must know in advance $P_t(\underline{x})$, p_t , $\phi_t(\underline{x})$ for $t = N, N-1, \dots, 1$ in order to solve the N-period problem. However, the results of Models Ia, Ib, II indicate that an explicit knowledge of $P_t(\underline{x})$, $\phi_t(\underline{x})$ for all t is not required a priori. Instead, at the *start of period* $t + 1$, the I.C. *must know* the following quantities:

[†]The issues discussed in this section are a simplified adaptation of a section in Hakansson (1969c) with some slight variations.

$P_{t+1}(\underline{x})$ the price function which is experienced in the Reinsurance Market;

$\phi_{t+1}(\underline{x})$ the probability density function of the risks of the companies participating in the Reinsurance Market. This can be either subjective or objective.

$\pi_t \dots \pi_1$ which is equivalent to knowledge of the interest rates for the next t periods (interest rate = $\frac{1 - \pi_t}{\pi_t}$) (possibly by forecasting).

$P_{t+1}\pi_{t+1} - \rho_{t+1}, \dots, P_1\pi_1 - \rho_1$ the Market value of the portfolio of policies (premiums minus losses) sold to customers of the I.C. (forecasting).

and

For Model Ia:

m_t, \dots, m_1 the value of a unit of risky asset (after reinsurance) for the next t periods (forecasting)

For Model Ib:

Nothing more is needed to calculate the optimal policies for reinsurance and dividend payments. The quantities, $q_t \dots q_1$ are only needed in calculating the value of the expected utility for a $t + 1$ -period problem.

For Model II:

w_t, \dots, w_1 the cost of a unit of risky asset (after reinsurance) for the next t -periods (forecasting).

A Simplifying Assumption

The data collection problem is simplified immensely if we assume

the market to be stationary in the following sense:

For all t

$$\phi_t(\underline{x}) = \phi(\underline{x})$$

$$P_t(\underline{x}) = \pi_t P(\underline{x}) \quad \text{where} \quad \int P(\underline{x}) d\underline{x} = 1$$

These assumptions imply that: (a) the number of companies remains the same in the reinsurance market, (b) the risks \underline{x}_t brought into the market by the participating companies are identically distributed from period to period, (c) the price function remains the same except for adjustments due to changes in the interest rate.

As a consequence of the above assumptions, we can now write:

$$\begin{aligned} \rho_t &= \pi_t \rho & ; & \quad \rho \equiv \int \underline{x} P(\underline{x}) d\underline{x} \\ m_t &= \pi_t^{\frac{c+1}{c}} m & ; & \quad m \equiv \int \left(\frac{P}{\phi} \right)^{1/c} P d\underline{x} \quad (\text{Model Ia}) \end{aligned}$$

$$q_t = q - \log \pi_t \quad ; \quad q \equiv E \log \frac{\phi}{P} \quad (\text{Model Ib})$$

$$w_t = \pi_t w + \pi_t \log \pi_t \quad ; \quad w \equiv \int \left(\log \frac{P}{\phi} \right) P d\underline{x} \quad (\text{Model II})$$

Thus the I.C. in this case needs to know or assess at the start of period t only π_t, \dots, π_1 and the premiums p_t, \dots, p_1 to be collected from its customers.

4.3 The Optimal Dividend Policy

For all models we found that $C_t^* = A_t R_t + B_t$ or that the dividends paid at the start of period $(t, t-1)$ due to results of period $(t+1, t)$ consist of a portion A_t of the reserves level R_t at the start of period $(t, t-1)$ plus a constant. (The transactions

between I.C. and stockholders will be explained in detail in Section 4.3.)

Because of the condition $aC_t^* + b > 0$ in Models Ia, Ib, we can make the following observations:

Model Ia

A. Model Ia₃ $c > 0, a < 0 \rightarrow C_t^* < -\frac{b}{a}$

(a) $b > 0 \rightarrow -\frac{b}{a} > 0$ thus C_t^* can be both *positive* or *negative*

(b) $b < 0 \rightarrow -\frac{b}{a} < 0$ thus C_t^* is *always negative*

B. Models Ia₁, Ia₂ $c < 0, a > 0 \rightarrow C_t^* > -\frac{b}{a}$

(a) $b > 0 \rightarrow -\frac{b}{a} < 0$ thus C_t^* can be both *positive* or *negative*

(b) $b < 0 \rightarrow -\frac{b}{a} > 0$ thus C_t^* is *always positive*

Cases A(a)(b) correspond to an increasing risk aversion utility function while B(a)(b) to decreasing risk aversion. It is thus questionable if Model Ia₃ can make sense in reality as it is rather improbable that individuals or firms would be increasing risk averse. If this were the case then the "elementary contracts" of the reinsurance market (or in general the securities) would be regarded as *inferior* goods since the amount of securities bought from the market would decrease with the wealth of the firm. If we decide that the above argument holds in the situation we are faced with, then only models Ia₁, Ia₂ make sense.

Allowing or restricting the dividends to be positive and (or) negative is also discussable. If we are faced with a situation

where nonnegative dividends are a "must" we must restrict to case B(b). It is doubtful, however, if such a situation applies to an insurance business, since it is plausible that stockholders faced with the prospects of high profits in future years will wish to save an insurance business from ruin by increasing the equity capital (negative dividends). (This point has been argued by Borch (1969).) In this sense, to allow the possibility of negative dividends is not a disadvantage but an advantage of the model; it simply means that the stockholders, once they have decided that they will stay in business for the next t periods, agree that an increase in the reserves of the I.C. will make it more profitable. In Section (4.7.2) we come back to this question and discuss a possible way of generalizing the problem to allow for a decision to liquidate the I.C. at any step. This would offer better ground for permitting dividends to become negative.

Model Ib

The utility function here has decreasing risk aversion. Also since $C_t^* > -\frac{b}{a}$, depending on the sign of b , C_t^* is either restricted to be positive or is allowed to take both positive and negative values. Similar remarks hold here are for Model Ia.

Also note that in this Model neither A_t nor B_t depend on $\phi_t(\underline{x})$ or $P_t(\underline{x})$ which says that the amount paid as dividend is independent of the favorableness or not of the risky asset

$$\frac{\phi_t(\underline{x})}{P_t(\underline{x})}.$$

Model II

The utility function has a constant risk aversion and as defined the Model itself imposes no restriction on C_t^* which is allowed to be positive or negative. If $C_t^* \geq 0$ must be included in the Model the same solution and formulae will hold once the following conditions are satisfied: (sufficient conditions)

- (1) The starting capital $R_N \geq -\frac{B_N}{A_N}$
- (2) $\log \alpha \geq \log \frac{P_t(\underline{x})}{\phi_t(\underline{x})}$ for all $\underline{x} \in X_t$, for $t = N, N-1, \dots, 1$

Certainly the first condition implies that $C_N^* \geq 0$. Suppose now

that $R_{t+1} \geq -\frac{B_{t+1}}{A_{t+1}}$ (or $C_{t+1}^* \geq 0$) we will show that condition

(2) implies that $C_t^* \geq 0$ or

$$R_t \geq -\frac{B_t}{A_t}.$$

The latter can be shown by substituting (3.61) in the above inequality and then using (3.56c). After some manipulations we obtain

$$\left(R_{t+1} + \frac{B_{t+1}}{A_{t+1}}\right) \left(\frac{1 - A_{t+1}}{\pi_{t+1}}\right) + \frac{\log \alpha}{\gamma A_t} \geq \frac{1}{A_t} \log \frac{P_{t+1}(\underline{x})}{\phi_{t+1}(\underline{x})}$$

and since the first term is nonnegative by assumption the result obtains. A necessary condition for condition (2) above to hold is $\pi_t \log \alpha \geq w_t \forall t$. The latter requires that $\alpha \geq \pi_t$. This can be seen once we observe that $w_t \geq \pi_t \log \pi_t$ for all t (see Section 4.5 inequality (4.9)).

4.4 The Optimal Reinsurance Treaty

Model Ia

We found that

$$(3.34) \quad R_t(\xi_{t+1}) = \frac{\lambda^{1/c}}{aA_t} \left(\frac{P_{t+1}(\xi_{t+1})}{\phi_{t+1}(\xi_{t+1})} \right)^{1/c} - \frac{b}{aA_t} - \frac{B_t}{A_t}$$

and looking at (3.44) for $\frac{\lambda^{1/c}}{aA_t}$ we see that the I.C. borrows the

amount $\frac{b}{aA_t} + \frac{B_t}{A_t} \pi_{t+1}$ (which grows to $\frac{b}{aA_t} + \frac{B_t}{A_t}$ by the end of

period $t+1$) from the market. The I.C. sells its portfolio of policies to the market and receives $P_{t+1}\pi_{t+1} - \rho_{t+1}$; then looks

at the cash it has on hand, $\left(R_{t+1} + P_{t+1}\pi_{t+1} - \rho_{t+1} + \frac{b\pi_{t+1}}{aA_t} + \frac{B_t}{A_t} \pi_{t+1} \right)$,

from which it gives A_{t+1} to the s/p-holders, while they pay back

to the I.C. $\frac{A_{t+1}}{A_t} \frac{bm_{t+1}}{a\lambda^{1/c}}$ which is the cost of buying $\frac{A_{t+1}}{A_t} \frac{b}{a\lambda^{1/c}}$

units of risky asset. Then the I.C. is left with

$$\begin{aligned} (1 - A_{t+1}) \left(R_{t+1} + P_{t+1}\pi_{t+1} - \rho_{t+1} + \frac{b\pi_{t+1}}{aA_t} + \frac{B_t}{A_t} \pi_{t+1} \right) \\ + \frac{A_{t+1}}{A_t} \frac{b}{a\lambda^{1/c}} m_{t+1} \end{aligned}$$

which is divided by m_{t+1} (the cost per unit risky asset) to find

how many units of $\left(\frac{P_{t+1}(\xi)}{\phi_{t+1}(\xi)} \right)^{1/c}$ are to be bought.

It is important to note that the choice of postreinsurance form of risky asset is independent of the prereinsurance wealth of the I.C. The latter only determines the amount to be bought.[†]

Now it is evident that since, $a(A_t R_t + B_t) + b > 0$ (this is assured if $a(A_{t+1} R_{t+1} + B_{t+1}) + b > 0$) we have

$$(4.1) \quad \text{if } a > 0 \quad R_t(\underline{\xi}) > -\frac{b}{aA_t} - \frac{B_t}{A_t} \quad \forall \underline{\xi}, t$$

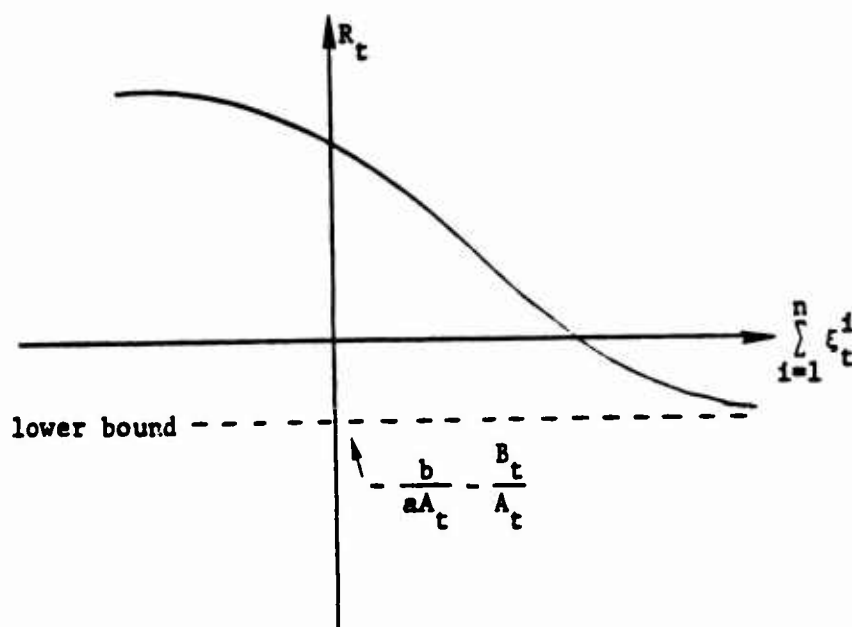


Figure 2a

The postreinsurance wealth as a function of the aggregate claims

$$\sum_{i=1}^n \xi_t^i \quad \text{for models Ia}_1, \text{Ia}_2 \quad (a > 0)$$

[†]This was expected since the LRT-class is closely related to the separation property of portfolio selection theories as we discussed in Section 3.4.1.

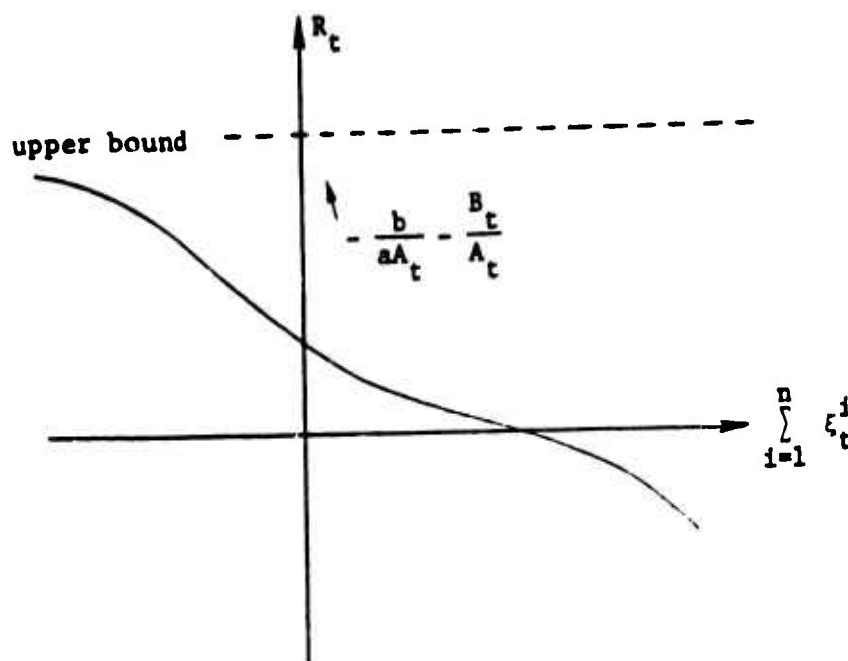


Figure 2b

The postreinsurance wealth as a function of the aggregate claims

$\sum_{i=1}^n \xi_t^1$ for model Ia_3 ($a < 0$)

$$(4.2) \quad \text{if } a < 0 \quad R_t(\underline{x}) < -\frac{b}{aA_t} - \frac{B_t}{A_t} \quad \forall \underline{x}, t$$

Since $\frac{P(\underline{x})}{\phi(\underline{x})}$ increases in each x^i when all the agents in the Market agree on the probability density function (recall Section 2.2.3/5 and that $y = \sum_{i=1}^n (C^i - x^i)$), we have the following cases:

A. If $a > 0, c < 0$ (Models Ia_1, Ia_2)

$\frac{P}{\phi}$ increases $\rightarrow \left(\frac{P}{\phi}\right)^{1/c}$ decreases thus $R_t(\underline{x})$ decreases in each x^i since $a > 0$;

B. If $a < 0, c > 0$ (Model Ia_3)

Then $R_t(\underline{x})$ again decreases in each x^1 since $a < 0$.

These observations are summarized in Figures 2a, 2b.

Model Ib

The quantities $\frac{b}{aA_t} + \frac{B_t}{A_t}$, $\frac{1}{aA_t\lambda}$ can be explained in a similar

way as in Model Ia, only here, the cost of a unit of investment

$\frac{\phi(\underline{x})}{P(\underline{x})}$ is $\int \frac{\phi}{P} P d\underline{x} = 1$. Thus $\frac{1}{aA_t\lambda}$ represents both the amount of money left to the I.C. after dividend payments and the number of units of risky asset $\frac{\phi}{P}$ bought.

Again, the prereinsurance total wealth enters in the formula for $R_t(\underline{x})$ only to determine the level of post-reinsurance risky asset and not its form.

As in Model Ia, because $\frac{P(\underline{x})}{\phi(\underline{x})}$ increases in each x^1 ,

$$(3.51) \quad R_t(\underline{x}) = \frac{1}{aA_t\lambda} \frac{\phi(\underline{x})}{P(\underline{x})} - \frac{b}{aA_t} - \frac{B_t}{A_t}$$

decreases in each x^1 and because of the condition

$$a(A_t R_t + B_t) + b > 0$$

R_t is bounded by (note $a > 0$ here)

$$(4.3) \quad R_t > -\frac{b}{aA_t} - \frac{B_t}{A_t}$$

These results are summarized in Figure 3.

Model II

We can explain the various quantities in (3.61) as follows:

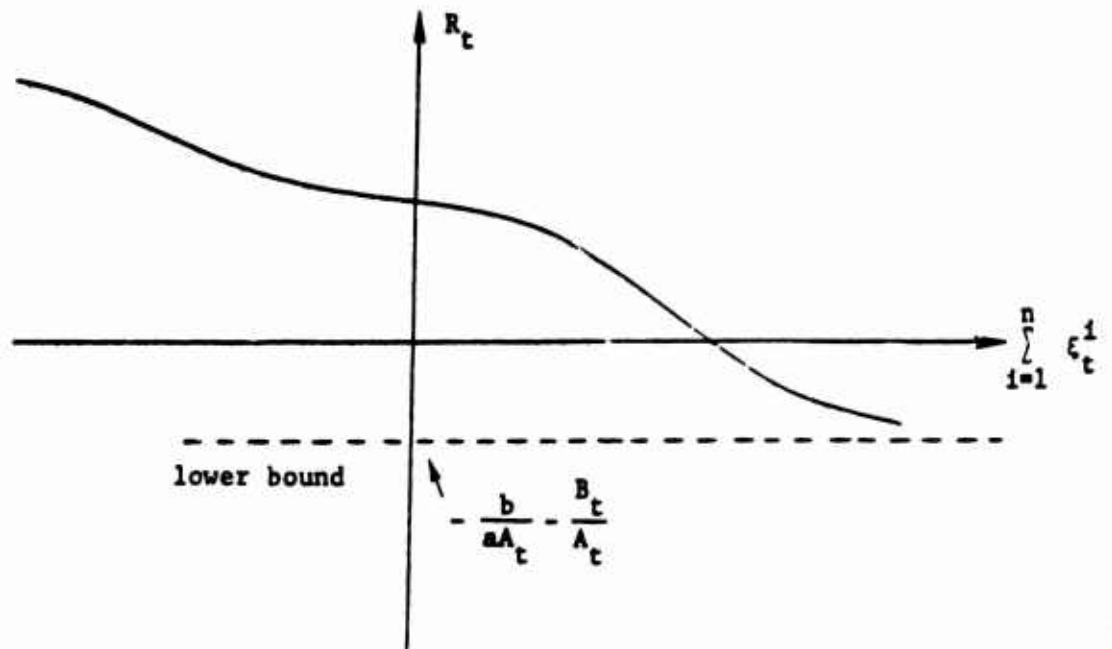


Figure 3

The postreinsurance wealth as a function of the aggregate

claims $\sum_{i=1}^n \xi_t^i$ for Model Ib.

The I.C. borrows from the market the amount $\left(\frac{A_{t+1}B_t}{A_t}\right)\pi_{t+1}$ which, by the end of the period, grows to $\frac{A_{t+1}B_t}{A_t}$ as debt. Then it sells all its risks and premiums, thus receiving the amount $p_{t+1}\pi_{t+1} - \rho_{t+1}$. It then assumes $\frac{1}{\gamma A_t}$ units of the risk $-\log \frac{P(\xi)}{\phi(\xi)}$ for which it receives a premium $\frac{w_{t+1}}{\gamma A_t}$ (recall w_{t+1} is the cost per unit of $\log \frac{P}{\phi}$). Next, the I.C. pays to the s/p-holders (look at 3.55 for C_{t+1}^*) the amount it borrowed from the market) $\frac{A_{t+1}B_t}{A_t}\pi_{t+1}$, and a portion A_{t+1} of $\left(R_{t+1} + p_{t+1}\pi_{t+1} - \rho_{t+1} + \frac{w_{t+1}}{\gamma A_t}\right)$, while the s/p-holders pay to the I.C. the fixed amount $\pi_{t+1} \frac{A_{t+1}}{\gamma A_t} \log a$. Finally, the I.C. places in the bank whatever cash it has left in its hands after these transactions. Thus, its value at the end of the period grows to $\frac{1}{\pi_{t+1}}$ of its original value.

It is interesting to note that the amount of risky asset the I.C. assumes is fixed, independent of its pre-reinsurance wealth. This is of course a well-known property of the exponential utility function.

The behavior of $R_t(\underline{x})$ w.r.t. each x^i is similar to the other models. Since $\frac{P(\underline{x})}{\phi(\underline{x})}$ increases in each x^i , $-\frac{1}{\gamma A_t} \log \frac{P(\underline{x})}{\phi(\underline{x})}$ decreases in each x^i and thus $R_t(\underline{x})$ decreases in each x^i as expected. See Figure 4.

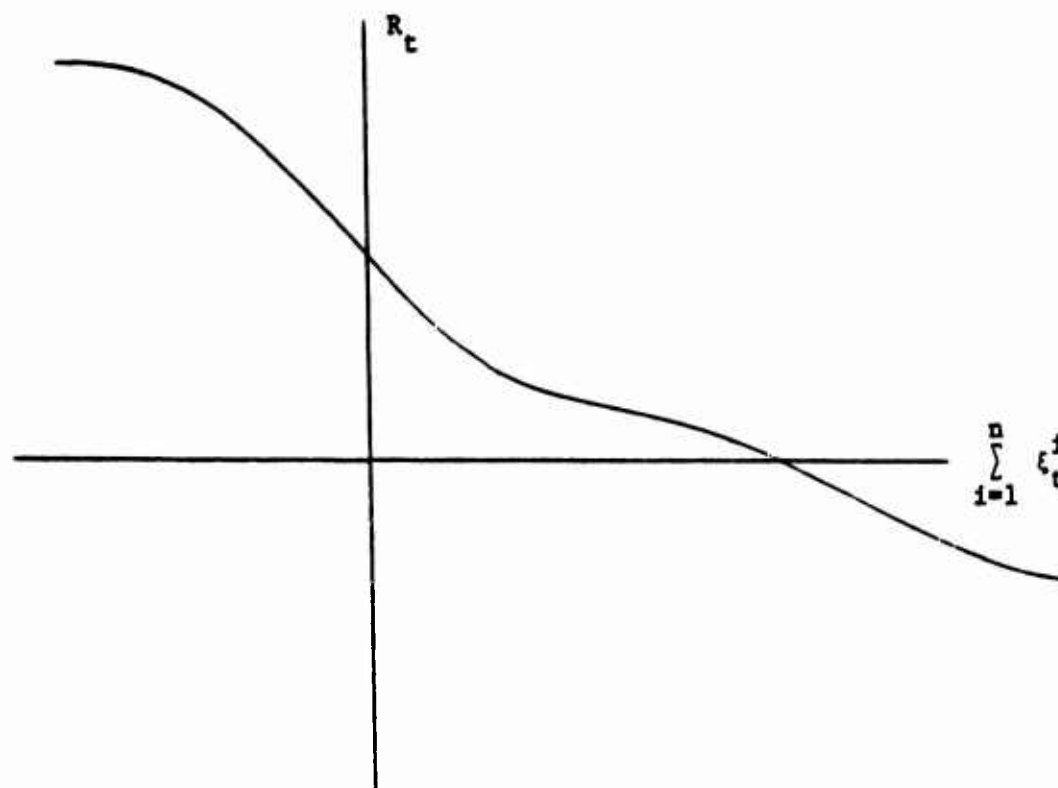


Figure 4

The postreinsurance wealth as a function of the aggregate
 claims $\sum_{i=1}^n \xi_t^i$ for Model II.

4.5 Inequalities between the Cost per Unit of Risky Asset, Interest and the Expected Value of Risky Asset

Model Ia

Suppose that we follow an optimal t -period program. Then

$$f_t(x) = D_t u(A_t x + B_t)$$

Now at the start of period $t + 1$ we first optimize w.r.t. $R_t(\cdot)$ keeping C_{t+1} fixed. Since we can always choose to sell all of our risky assets and still satisfy the budget constraint, the optimizing value of R_t , R_t^* must be such that

$$(4.4) \quad u\left(A_t \frac{y}{\pi_{t+1}} + B_t\right) \leq Eu[A_t R_t^* + B_t]$$

where

$$y \equiv \int \left(\frac{R_{t+1} - C_{t+1}}{\pi_{t+1}} + p_{t+1} - x_{t+1} \right) p_{t+1}(x) dx.$$

Certainly choosing $R_t = \frac{y}{\pi_{t+1}}$ is feasible, since we satisfy the budget constraint, $\int R_t p dx = y$, and also $a\left(A_t \frac{y}{\pi_{t+1}} + B_t\right) + b > 0$ because of (3.36). Now using (3.34) for R_t^* in (4.4),

$$\begin{aligned} u\left(A_t \frac{y}{\pi_{t+1}} + B_t\right) &\leq E u\left(\frac{\lambda^{1/c}}{a} \left(\frac{p_{t+1}}{\phi_{t+1}}\right)^{1/c} - \frac{b}{a}\right) \\ &\leq \frac{1}{a(c+1)} \lambda^{c+1/c} m_{t+1} \end{aligned}$$

using (3.35) for $\lambda^{1/c}$, the last inequality can be replaced by

$$\leq \frac{m_{t+1}}{a(c+1)} \left[\frac{aA_t}{m_{t+1}} \underbrace{(R_{t+1} - C_{t+1} + p_{t+1} \pi_{t+1} \rho_{t+1})}_y + \frac{b}{m_{t+1}} \pi_{t+1} + \frac{B_t \pi_{t+1}^a}{m_{t+1}} \right]^{c+1}$$

and rearranging

$$u\left(A_t \frac{y}{\pi_{t+1}} + B_t\right) \leq m_{t+1}^{-c} \pi_{t+1}^{c+1} u\left(A_t \frac{y}{\pi_{t+1}} + B_t\right).$$

Considering now all possible cases

$$a(c+1) \leq 0, \quad c \leq 0, \quad ac < 0$$

and using the form of $u(\cdot)$ for Model Ia we get

$$(4.5) \quad \left. \begin{array}{l} a > 0, \quad -1 < c < 0 \\ \text{or} \\ a < 0, \quad c > 0 \end{array} \right\} \leftrightarrow m_{t+1} \leq \pi_{t+1}^{c+1/c} \quad ; \quad N-1 \leq t \leq 0$$

We can follow a similar approach using Jensen's inequality

$$f_t(ER_t) > Ef_t(R_t),$$

since f_t is strictly concave.

$$\text{Now let } \eta_{t+1} \equiv E \left(\frac{p_{t+1}(\xi_{t+1})}{\phi_{t+1}(\xi_{t+1})} \right)^{1/c} = \text{the expected value of a unit}$$

of risky asset. Then

$$u\left[\frac{\lambda^{1/c}}{a} \eta_{t+1} - \frac{b}{a}\right] > Eu(R_t)$$

and by manipulating we obtain

$$\frac{\eta_{t+1}^{c+1}}{a(c+1)} > \frac{m_{t+1}}{a(c+1)} .$$

Again, exhausting the possibilities, we have

$$(4.6) \quad \begin{aligned} a(c+1) < 0 &\rightarrow \eta_{t+1}^{c+1} < m_{t+1} \\ a(c+1) > 0 &\rightarrow \eta_{t+1}^{c+1} > m_{t+1} \end{aligned} \quad N-1 \leq t \leq 0 .$$

Summarizing, since $ac < 0$, there are three possibilities:

$$(4.7) \quad \begin{aligned} \text{Model Ia}_1: & \quad a > 0, c < -1 \rightarrow \eta_t^{c+1} < m_t \leq \pi_t^{c+1/c} \quad (i) \\ \text{Model Ia}_2: & \quad a > 0, -1 < c < 0 \rightarrow \eta_t^{c+1} > m_t \geq \pi_t^{c+1/c} \quad (ii) \\ \text{Model Ia}_3: & \quad a < 0, c > 0 \rightarrow \eta_t^{c+1} < m_t, m_t \geq \pi_t^{c+1/c} \quad (iii) \end{aligned}$$

for $N \leq t \leq 1$.

Model Ib

Here we can only say that

$$(4.8) \quad q_t > -\log \pi_t$$

where

$$q_t \equiv E \log \frac{\phi_t(\xi)}{P_t(\xi)} ; \quad N \leq t \leq 1$$

this follows easily by Jensen's inequality.

Model II

Here we have already defined

$$(3.58) \quad w_{t+1} \equiv \int \log \left(\frac{P_{t+1}}{\phi_{t+1}} \right) P_{t+1}$$

w_{t+1} is the cost of a unit of risky asset.

Similarly to Model Ia, we argue that

$$\begin{aligned} u \left(A_t \frac{y}{\pi} + B_t \right) &\leq E u(A_t R_t^* + B_t) \\ &\leq E u \left(-\frac{1}{\gamma} \log \lambda \frac{P_{t+1}}{\phi_{t+1}} \right) \\ &\leq \frac{1}{\gamma} (1 - \lambda \pi_{t+1}) \end{aligned}$$

using (3.60) for λ we have

$$\begin{aligned} e^{-\left(A_t \frac{y}{\pi_{t+1}} + B_t \right)} \left(\frac{\pi_{t+1} e^{-w_{t+1}/\pi_{t+1}}}{\gamma} - \frac{1}{\gamma} \right) &\leq 0 \\ \pi_{t+1} e^{-w_{t+1}/\pi_{t+1}} &\leq 1 \quad ; \quad N-1 \leq t \leq 0 \end{aligned}$$

$$(4.9) \quad \text{or} \quad -\frac{w_t}{\pi_t} \leq \log \frac{1}{\pi_t} \quad ; \quad N \leq t \leq 1$$

which says that cost of a unit of risky asset is greater than $\pi \log \pi$ in each period.

Finally, using Jensen's inequality

$$(4.10) \quad E \log \frac{P_t}{\phi_t} < \log \pi_t \quad ; \quad N \leq t \leq 1$$

which says that the expected value of a unit of risky asset is less than $\log \pi$ in each period.

4.6 The Infinite Horizon Stationary Case

Let $p_t = p$, $P_t(\underline{x}) = P(\underline{x})$, $\phi_t(\underline{x}) = \phi(\underline{x})$ for all t . Then our formulae for all the models examined in Chapter 3 can be extended to the infinite horizon case by letting $t \rightarrow \infty$.

We will denote

$$\lim_{t \rightarrow \infty} (D_t, A_t, B_t, E_t) = (D, A, B, E)$$

Model Ia

Condition $L < 1$ where $L \equiv \frac{m}{a(1/c)}$, $m \equiv \int \left(\frac{P(\underline{x})}{\phi(\underline{x})} \right) P(\underline{x}) d\underline{x}$

$$(4.11a) \quad D = \frac{1}{1 - L}$$

$$(4.11b) \quad A = 1 - L$$

$$(4.11c) \quad B = \frac{1 - L}{1 - \pi} (p\pi - \rho) + \frac{b}{a(1 - \pi)} (\pi - L)$$

$$(4.12) \quad f(R) = D u(AR + B)$$

and we can check that in fact this satisfies

$$f(R) = \max_{C, R(\cdot)} \{u(C) + \alpha E f[R(\xi)]\}$$

$$\text{s.t.} \quad \int R(\underline{x}) P(\underline{x}) d\underline{x} = \int \left(\frac{C - x}{\pi} + p - x \right) P(\underline{x}) d\underline{x}$$

where $C = AR + B$, and $a(AR + B) + b > 0$.

Further, since for each t $A_t, D_t, B_t, R_t, C_t^*$ are unique, the solution is unique in the limit.

Model IbCondition $\alpha < 1$

$$(4.13a) \quad D = \frac{1}{1-\alpha}$$

$$(4.13b) \quad A = 1-\alpha$$

$$(4.13c) \quad B = \frac{1-\alpha}{1-\pi} (p\pi-\rho) + \frac{b}{a(1-\pi)} (\pi-\alpha)$$

$$(4.13d) \quad E = \frac{\alpha}{a(1-\alpha)^2} [\log \alpha + q], \quad q \equiv E \log \left(\frac{\phi}{p} \right)$$

$$f(R) = D u(AR + B) + E$$

$$C^* = AR + B$$

Model IICondition $\pi < 1$

$$(4.14a) \quad D = \frac{1}{1-\pi}$$

$$(4.14b) \quad A = 1-\pi$$

$$(4.14c) \quad B = p\pi-\rho + \frac{w}{\gamma(1-\pi)} - \frac{\pi}{\gamma(1-\pi)} \log \alpha, \quad w = \int \log \left(\frac{p}{\phi} \right) p d\mathbf{x}$$

$$(4.14d) \quad E = \frac{1}{\gamma(1-\pi)} \frac{\alpha-\pi}{1-\alpha}$$

$$f(R) = D u(AR + B) + E$$

$$C^* = AR + B$$

4.7 Generalizations4.7.1 Simple Generalizations - Variations

It is easy to see that we can allow the patience factor α

to depend on time. We only need to replace α by α_t in the results of Chapter 3.

Another simple extension refers to the market mechanism. The I.C. is not concerned with the number of companies participating in the market or whether that number changes with time as long as an equilibrium price function exists which is known or forecasted for the next N -periods. As a matter of fact, the approach in Chapter 3 would be the same if the I.C. in question was not faced with a Reinsurance Market but with a reinsurer quoting prices for any type of contract. The latter situation would be better described in this way:

1) The I.C. announces its intention in a certain period to reinsure its portfolio which is subject to the claims r.v. ξ with known p.d.f. $\phi(x)$ (i.e., it wishes to form a contract as a function of x , the value of the r.v. ξ).

2) The reinsurer based on the p.d.f. $\phi(x)$ quotes a price function $P(x)$ such that the price of any reinsurance contract $Z(x)$ will be

$$\int Z(x)P(x)dx$$

d) The I.C., faced with $P(x)$, finds the best contract $Z(x)$ to fit its needs subject to a budget constraint

$$\int Z(x)P(x)dx = \int (C - x)P(x)dx$$

where C is the prior capital of the I.C., $C - \xi$ is the pre-reinsurance wealth of the I.C., and $Z(x)$ is the post-reinsurance

wealth of the I.C. for the beginning of the next period as a function of the realization x of the claims r.v. ξ .

This situation is mathematically the same as that of Chapter 3 and the same results hold. As a side comment, we note that this way of calculating premiums (through a price function) could be used also in standard insurance. We discuss this possibility in Appendix II.

Finally, we remark that small variations in the budget constraint (2.2) will still give us closed form results. For example, the premiums p_t might be collected gradually and not at the end of the period. This would change the prereinsurance value of the

I.C. from $\frac{R_t - C_t}{\pi_t} + p_t - \xi_t$ to $\frac{R_t - C_t}{\pi_t} + \frac{p_t}{\pi'_t} - \xi_t$; where

$\pi'_t \geq \pi_t$; where π'_t can be calculated on the basis of an averaged interest rate. Such variations are also acceptable for C_t or ξ_t .

4.7.2 The Right to Liquidate the I.C.

The problem as we developed it in Chapter 3 finds the optimal policies when the I.C. is determined to stay in business for N -periods. But this approach does not permit the shareholders to liquidate the I.C. at any period, which may be desirable, especially when dividends can be negative. To resolve this problem we need to know how the shareholders trade off one more period of operation compared to liquidating the I.C.

First, to be consistent we must talk about a fixed interval, say of N -periods. Then facing those N -periods, the management should decide on one of the following alternatives

- 0) Liquidate now
- 1) Operate one period then liquidate

2) Operate two periods then liquidate

.

.

.

N) Operate N periods

But to evaluate each alternative, the management must attach a value to the duration of time the company does not exist. For example, let the starting capital be R , then the k th alternative says "operate for k periods and be out of existence for the rest $N-k$ periods." We want to attach a value G_{N-k} to the $N-k$ periods of nonexistence of the I.C.

At first thought, G_j should correspond to j periods of $u(0)$. But this is not correct as the shareholders will have incomes from other sources. An indirect way of determining G_j would be to think of it as j periods of $u(C^0)$ where C^0 is the minimum dividend acceptable by the shareholders if they were to receive a constant dividend C^0 for each of j periods of operation of the I.C. Below C^0 the shareholders decide to liquidate. In effect, we say $G_j = \sum_{t=1}^j \alpha^t u(C^0)$ for C^0 as described above. Still this explanation for G_j is not entirely satisfactory, as in general G_j should depend on the terminal capital R ; the capital of the I.C. at the moment of liquidation.

However, if we rather simplistically accept that we know $G_1 \dots G_N$ and assume that they are independent of the terminal capital, we have a measure of comparing periods with the I.C. in operation versus periods after liquidation. Our problem then is reduced to finding the maximum of

$$u(R) + G_N, f_1(R) + \alpha G_{N-1}, \dots, f_{N-1}(R) + \alpha^{N-1} G_1, f_N(R)$$

where, as before, $f_k(R)$ is the maximum expected utility for k periods of operation starting now with capital R .

In particular, when u belongs to the LRT class $f_1(R), \dots, f_N(R)$ can be calculated by using the recursive formulas found in Chapter 3.

In actual operation the I.C. is faced with an *indefinite* horizon where the N -periods have only the meaning of a *planning ahead* interval. In other words, at the start of each period of the indefinite horizon the I.C. will revise and expand its forecasts to cover the N coming periods and will then solve an N -alternative problem as discussed in this Section. Thus, liquidation will actually occur when $u(R) + G_N$ is the maximum valued alternative.

4.7.3 Decision on Amount to be Spent for Promotion of Sales

Suppose that the I.C., apart from deciding on the optimal dividend payments and reinsurance treaty, has to decide on how much money to allocate to promote its volume of business. It turns out that this extra decision does not alter the optimal policies found in the above models.

We keep the same notation except that now $p_t - \xi_t$ is replaced by $v_t \zeta$ where

ζ a random variable representing the typical contract the I.C. forms with its customers. Its value will be z .

For simplicity of exposition, we assume ζ to be independent, identically distributed in each period.

v_t the volume of business in period t (the amount of ζ).

r_t amount of money allocated to sales promotion in period t
(we assume the effect of this allocation to appear in the

volume of the next period, $t - 1$).

We further suppose that

$$(4.15) \quad v_t = hv_{t+1} + g(r_{t+1})$$

where $0 < h \leq 1$, to account for a drift downwards of sales without sales effort, and $g(r)$ a concave, continuously differentiable, increasing positive function on $0 \leq r < \infty$ with

$$(4.16) \quad g'(0) \geq \frac{1}{s\pi}^{\dagger}$$

where $s \equiv \int zP(z)dz$ the value of a typical contract ζ in the reinsurance market. Condition (4.16) rewritten as $sg'(0) \geq \frac{1}{\pi}$, says "The return $sg'(0)$ brought by an initial \$1 spent for promotion of sales must be greater than the return $\frac{1}{\pi}$ from a \$1 left to mature over a period." This is a sufficient condition for the problem to have a solution but even a milder condition would do, as we will see later (4.25).

The problem can again be formulated by a Dynamic Programming relation

$$(4.17) \quad f_{t+1}(R_{t+1} \mid v_{t+1}) = \max_{\substack{C_{t+1}, R_t(\cdot) \\ r_{t+1} \geq 0}} [u(c_{t+1} + \alpha E f_t(R_t(\zeta) \mid y_t))]$$

such that

[†]For convenience we assume s, π independent of time. However, the results in this section can easily be adapted to the nonstationary case $s_t, \pi_t, h_t, g_t(\cdot), \zeta_t$.

$$\begin{aligned}
 & \int \left(\frac{R_{t+1} + C_{t+1} - r_{t+1}}{\pi} + v_{t+1} z \right) P(z) dz = \\
 (4.18) \quad & = \int R_t(\underline{z}) P(\underline{z}) d\underline{z}
 \end{aligned}$$

the boundary condition is

$$(4.19) \quad f_0(R_0 | v_0) \equiv u(R_0) .$$

Assuming $u(\cdot)$ to be one of the Models Ia, Ib, II we follow similar procedures. Namely, we fix r_t and maximize, as before, first w.r.t. $R_{t-1}(\underline{z})$ and then w.r.t. C_t the result prior to max w.r.t. r_t is

$$(4.20) \quad f_t(R_t | v_t) = \max_{r_t \geq 0} D_t u(A_t R_t + B_t) + E_t$$

$$(E_t = 0 \text{ for Model Ia})$$

and now D_t and A_t are not functions of r_t but are the same as in the original problem, while

$$(4.21) \quad B_t = A_t \left[s v_t - r_t + \frac{B_{t-1}}{A_{t-1}} \pi + \frac{b\pi}{aA_{t-1}} - \frac{b}{aA_{t-1}} \frac{m_t}{\alpha^{1/c}} \right] \text{ Model Ia}$$

$$(4.22) \quad B_t = A_t \left[s v_t - r_t + \frac{B_{t+1}}{A_{t-1}} \pi + \frac{b\pi}{aA_{t-1}} \frac{ba}{aA_{t-1}} \right] \text{ Model Ib}$$

$$(4.23) \quad B_t = A_t \left[s v_t - r_t + \frac{B_{t+1}}{A_{t+1}} \pi + \frac{w_t}{\gamma A_{t-1}} - \frac{\pi \log \alpha}{\gamma A_{t-1}} \right] \text{ Model II}$$

where $B_0 = 0$, $A_0 = 1$. Thus it only remains to maximize B_t w.r.t. r_t . The result is the same for all models so we will deal with Model Ia only. Expanding $\frac{B_t}{A_t}$ recursively we get:

$$\begin{aligned} \frac{B_t}{A_t} = & s(v_t + v_{t-1}\pi + \dots + v_1\pi^{t-1}) - (r_t + r_{t-1}\pi + \dots + \pi^{t-1}r_1) \quad t \geq 1 \\ (4.24) \quad & + \frac{b}{a} \delta \left[\frac{1}{A_{t-1}} + \frac{\pi}{A_{t-2}} + \dots + \frac{\pi^{t-1}}{A_0} \right] \end{aligned}$$

(r_1^* and r_0^* can be shown to be zero since the I.C. would spend no money in period 1 or 0 for the volume to increase after it ceases operation.) where we have set

$$\delta \equiv \pi - \frac{m}{a^{1/c}}$$

Using now

$$(4.15) \quad v_j = hv_{j+1} + g(r_{j+1}) \quad j \geq t$$

the first term in the expansion of $\frac{B_t}{A_t}$ (4.24) becomes:

$$\begin{aligned} & s(v_t + \pi[hv_t + g(r_t)]) + \pi^2[h^2v_t + hg(r_t) + g(r_{t-1})] + \dots \\ & \dots + \pi^{t-1}[h^{t-1}v_t + h^{t-2}g(r_t) + h^{t-3}g(r_{t-2}) + \dots + hg(r_3) + g(r_2)] = \\ & = s\{v_t(1 + \pi h + \pi^2 h^2 + \dots + \pi^{t-1} h^{t-1}) + g(r_t)(\pi + \pi^2 h + \pi^3 h^2 + \dots \\ & + \pi^{t-1} h^{t-2}) + g(r_{t-1})(\quad) + \dots + g(r_2)\} . \end{aligned}$$

Now $\frac{B_t}{A_t}$ is concave in r_t and thus differentiating (4.24) using the above expansion we obtain

$$\frac{d}{dz_t} \left(\frac{B_t}{A_t} \right) = sg'(r_t) \pi \sum_{k=0}^{t-2} (\pi h)^k - 1 \quad ; \quad t \geq 2 .$$

So equating to zero and solving, the optimal amount r_t^* to be spent for promotion of the sales volume satisfies

$$(4.25) \quad g'(r_t^*) = \frac{1}{s\pi \sum_{k=0}^{t-2} (\pi h)^k} ; \quad t \geq 2 .$$

The solution, $r_t^* \geq 0$, exists since we assumed that

$$g'(0) \geq \frac{1}{s\pi} \quad \text{and since} \quad \frac{1}{s\pi} \geq \frac{1}{s\pi \sum_{k=0}^{t-2} (\pi h)^k}$$

while $g'(\cdot)$ decreases continuously on the positive semi-axis by assumption.

(4.25) can be interpreted as follows: increase r_t until

$sg(r_t) \sum_{k=0}^{t-2} (\pi h)^k$, the marginal business discounted at its present

value brought by an extra dollar spent in this period, equals $\frac{1}{\pi}$

which is the value of that dollar when left to mature till the end of the period.

Summarizing

The N period problem examined here is essentially the same as the original problem. However, here the p_t 's are not given and we first have to determine the r_t^* 's by solving Equation (4.25). We then determine the v_t^* 's by using $v_{t-1}^* = hv_t + g(r_t^*)$ and an initial condition ($v_N^* = 0$, say). Finally, A_t, D_t, E_t can be determined by using the recurrence relations of the original problem

(Chapter 3). $\frac{B_t}{A_t}$ is then calculated by using (4.21), (4.22), (4.23) and the optimal r_t^*, v_t^* which are given by the same formula (4.25) for all models. Also the optimal reinsurance treaty of the original

problem remains the same; only replace $p_t \pi_t - \rho_t$ by $sv_t^* - r_t^*$.

In the infinite horizon case (4.25) becomes

$$(4.26) \quad g'(r^*) = \frac{1 - h\pi}{s\pi}$$

and for this to have meaning we could use a weaker assumption than $g'(0) \geq \frac{1}{s\pi}$, namely, $g'(0) \geq \frac{1 - h\pi}{s\pi}$. The formulae for the overall problem again are the same except for B where $p\pi - \rho$ must be replaced by $sv^* - r^*$. For example for Model Ia (4.11c) must be replaced by

$$(4.29) \quad B = \frac{1 - L}{1 - \pi} (sv^* - r^*) + \frac{b}{a} \frac{\pi - L}{1 - \pi}$$

where

$$(4.28) \quad v^* = hv^* + g(r^*) \rightarrow v^* = \frac{g(r^*)}{1 - h}$$

where r^* is the solution to (4.26) and is unique if $g'(z)$ is strictly decreasing. Also the formulas for the optimal reinsurance treaty must be changed by replacing $p_t \pi_t - \rho_t$ by $sv_t^* - r_t^*$.

CHAPTER 5

AN EXTENSION TO MULTIPLICATIVE UTILITIES

5.1 Introduction

Meyer (1969) has shown that when an individual decides on his future consumption independently of his past, the utility function evaluating possible consumption streams must be one of three forms (5.2), (5.3), (5.4). Form (5.2) is the additive form which we have already examined in Chapter 3. The other two, which we will call multiplicative forms, are the subject of this chapter.

We formulate once more the dynamic decision problem of the I.C. and proceed to find closed form solutions. Unfortunately, this time we are able to do so only for those utility functions that belong to the subclasses Ia_1 or Ia_2 . The results are strikingly similar. The dividend payments are linear in the reserves level and the reinsurance treaty is virtually the same as before. We are thus able to interpret the results in a manner similar to that of Chapter 4. The generalizations of Chapter 4 easily adapt to the present situation.

5.2 The Multiplicative Utility Function for Evaluating Consumption Streams

Let $\underline{C} = [C_1, \dots, C_N]$ be a vector of consumption (dividends) for N -periods. At any $1 \leq t \leq N$ we will denote

$$\vec{C}_t = [C_t, C_{t+1}, \dots, C_N] \quad (\text{the future consumption})$$

$$\vec{C}_t = [C_1, C_2, \dots, C_t] \quad (\text{the past consumption})$$

Consider an individual who evaluates at time t the value of consumption stream \vec{C}_{t+1} , given his past consumption \vec{C}_t , using

a utility function $u_t(\vec{C}_{t+1} | \vec{C}_t)$. Let also $u(\underline{C})$ denote his utility of the overall nonconditional stream at time $t = 0$.

Meyer (1969) defines \vec{C}_{t+1} as cardinal independent (c.i.) of \vec{C}_t if

$$(5.1) \quad u_t(\vec{C}_{t+1} | \vec{C}_t) = a(\vec{C}_t) + b(\vec{C}_t)u_t(\vec{C}_{t+1})$$

where $u_t(\vec{C}_{t+1})$ is a utility function evaluating consumption streams \vec{C}_{t+1} independently of \vec{C}_t . Certainly (5.1) implies that the individual will preserve the preference ordering of all possible \vec{C}_{t+1} for different \vec{C}_t . Meyer (1969) then shows that if \vec{C}_{t+1} is c.i. \vec{C}_t for $t = 1, \dots, N-1$ and \vec{C}_{N-1} c.i. of C_N , then $u(\underline{C})$ must be one of the three forms:

$$(5.2) \quad u(\underline{C}) = \sum_{t=1}^N u_t(C_t) \quad (i)$$

$$(5.3) \quad u(\underline{C}) = \prod_{t=1}^N u_t(C_t) ; u_t(\cdot) > 0, \text{ for all } t \quad (ii)$$

$$(5.4) \quad u(\underline{C}) = - \prod_{t=1}^N (-u_t(C_t)) ; u_t(\cdot) < 0, \text{ for all } t \quad (iii)$$

These results have been extended by Richard (1972). Expression (5.2), along with the assumption of stationarity, $u_t(\cdot) = \alpha^t u(\cdot)$, reduces to the utility function we used in expressing the I.C. problem in Chapter 3. We propose now to employ the multiplicative forms (5.3) and (5.4) in formulating the dynamic decision problem of the I.C. We will again find closed form results but only for Models Ia₁ and Ia₂.

5.3 Formulation of the Dynamic Decision Problem of the I.C. under a Multiplicative Utility Function - Closed Form Solutions for Models Ia₁, Ia₂.

Keeping exactly the same notation as in Chapter 3 and assuming that $u_t(\cdot) = u(\cdot)$ for all t , the Dynamic Programming (D.P.) relation that represents the I.C.'s decision problem can now be expressed for $u(\cdot) > 0$, using (5.3), as

$$(5.5) \quad f_{t+1}(R_{t+1}) = \max_{C_{t+1}, R_t(\cdot)} \left\{ u(C_{t+1}) E f_t[R_t(\xi_{t+1})] \right\}$$

along with the same budget constraint

$$(5.6) \quad \int R_t(x_{t+1}) P_{t+1}(x_{t+1}) dx_{t+1} = \int \left(\frac{R_{t+1} - C_{t+1}}{\pi_{t+1}} + p_{t+1} - x_{t+1}^1 \right) P_{t+1}(x_{t+1}) dx_{t+1}$$

and boundary condition

$$(5.7) \quad f_0(R_0) = u(R_0)$$

Similarly, for $u(\cdot) < 0$ using (5.4)

$$(5.8) \quad f_{t+1}(R_{t+1}) = \max_{C_{t+1}, R_t(\cdot)} \left\{ -u(C_{t+1}) E f_t(R_t) \right\}$$

along with the same budget constraint and boundary condition.

In the following, we will find closed form results from (5.5) when $u(\cdot) \in Ia_2$, and from (5.8) when $u(\cdot) \in Ia_1$.

5.3.1 Model Ia₁

One-period problem

Expression (5.8) becomes:

$$(5.9) \quad f_1(R_1) = \max_{C_1, R_0(\cdot)} [-u(C_1)E u(R_0(\xi_{t+1}))]$$

subject to

$$(5.10) \quad \int R_0(\underline{x}_1) P_1(\underline{x}_1) d\underline{x}_1 = \int \left(\frac{R_1 - C_1}{\pi_1} + p_1 - x_1^1 \right) P_1(\underline{x}_1) d\underline{x}_1$$

Fix C_1 .

Since $-u(C_1) > 0$ we only need to maximize $Eu(R_0)$ over $R_0(\cdot)$ given the budget constraint. Thus, we use the Market optimality condition (2.8):

$$(5.11) \quad u'(R_0) \phi_1 = \lambda P_1$$

for $u \in I_{A_1}$. Solving (5.11) we find the optimal postreinsurance wealth as a function of the risks \underline{x}_1 :

$$(5.12) \quad R_0(\underline{x}_1) = \frac{\lambda^{1/c}}{a} \frac{P_1(\underline{x}_1)^{1/c}}{\phi_1(\underline{x}_1)} - \frac{b}{a}$$

and substituting in the budget constraint we find

$$(5.13) \quad \lambda^{1/c} = \frac{a}{m_1} \left(R_1 - C_1 + p_1 \pi_1 - \rho_1 + \frac{b}{a} \pi_1 \right)$$

where m_1 , ρ_1 are as defined in Chapter 3. Again, because we require $\lambda^{1/c} > 0$ for (5.11) to have a solution (5.12), it follows that, we must require the condition:

$$(5.14) \quad a(R_1 - C_1 + p_1 \pi_1 - \rho_1) + b \pi_1 > 0$$

Substituting (5.12) and (5.13) in (5.9) we obtain

$$(5.15) \quad f_1(R_1) = \max_{C_1} \left\{ \frac{-m_1}{a(c+1)} u(C_1) \left(\frac{a}{m_1} Q_1 \right)^{c+1} \right\}$$

where

$$(5.16) \quad Q_1 \equiv R_1 - C_1 + p_1 \pi_1 - \rho_1 + \frac{b}{a} \pi_1$$

Now let $K = \frac{-m_1}{a(c+1)}$, $K > 0$ because $m_1 > 0$ and $a(c+1) < 0$.

Thus, we have to maximize

$$(5.17) \quad u(C_1) \left(\frac{a}{m_1} Q_1 \right)^{c+1}$$

over C_1 .

We take the first derivative of (5.17) w.r.t. C_1 :

$$(5.18) \quad \Delta_1 = u'(C_1) \left(\frac{a}{m_1} Q_1 \right)^{c+1} - u(C_1) \left(\frac{a}{m_1} Q_1 \right)^c (c+1) \frac{a}{m_1}$$

The second derivative w.r.t. C_1 ,

$$(5.19) \quad \begin{aligned} \Delta_2 = & u''(C_1) \left(\frac{a}{m_1} Q_1 \right)^{c+1} - 2u'(C_1) \left(\frac{a}{m_1} Q_1 \right)^c \frac{a(c+1)}{m_1} \\ & + u(C_1) \left(\frac{a}{m_1} Q_1 \right)^{c-1} (c+1)c \frac{a^2}{m_1^2} \end{aligned}$$

is negative if

$$(1) \quad \frac{a}{m_1} Q_1 > 0 \quad (\text{that is if (5.14) holds) and}$$

(ii) $aC_1 + b > 0$ (that is if $u(C_1)$ has meaning)

Substituting for $u(\cdot)$ in (5.19) becomes:

$$-\Delta_2 = \underbrace{\left(\frac{a}{m_1} Q_1\right)^{c-1} (aC_1 + b)^{c-1}}_{K_1} \left[- (ac) \left(\frac{a}{m_1} Q_1\right)^2 \right. \\ \left. + 2(aC_1 + b) \left(\frac{a}{m_1} Q_1\right) (c+1) \frac{a}{m_1} + (aC_1 + b)^2 \frac{(-ac)}{m_1^2} \right]$$

$K_1 > 0$ because of (i) and (ii)

Now since $-ac > 0$ we can write:

$$-\Delta_2 = K_1 \left[\left(\frac{a}{m_1} Q_1 \sqrt{-ac} - \frac{(aC_1 + b)}{m_1} \sqrt{-ac} \right)^2 \right. \\ \left. + 2(aC_1 + b) \frac{a^2}{m_1^2} Q_1 \right]$$

and since $aC_1 + b > 0$, $\frac{a}{m_1} Q_1 > 0$, $a > 0$ for Model Ia₁ it follows

follows that $\Delta_2 < 0$.

Now setting $\Delta_1 = 0$ we are guaranteed of a unique maximum as long as (i) and (ii) above hold. After some simplifications we find that the optimal dividend C_1^* should satisfy

$$(5.20) \quad aC_1^* + b = aQ_1^*$$

where

$$Q_1^* \equiv Q_1(C_1^*)$$

or using the definition of Q_1 in (5.16)

$$(5.21) \quad C_1^* = \frac{1}{2} \left(R_1 + p_1 \pi_1 - \rho_1 + \frac{b}{a} [\pi_1 - 1] \right)$$

which is linear in R_1 and does not depend on M_1 - the cost of a unit of risky asset, $\left(\frac{P_1(\underline{x}_1)}{\phi_1(\underline{x}_1)}\right)^{1/c}$. Conditions (i) and (ii) are now equivalent because of (5.20). Thus we only require:

$$(5.22) \quad aC_1^* + b > 0$$

Substituting (5.20) back in (5.15) we find that the maximum expected utility for the one period problem starting with reserves R is:

$$f_1(R_1) = -u(C_1^*) \frac{m_1}{a(c+1)} \left(\frac{aC_1^* + b}{m_1} \right)^{c+1}$$

or

$$(5.23) \quad \begin{aligned} f_1(R_1) &= -\frac{1}{m_1^c} u(C_1^*)^2 \\ &= -D_1 u(A_1 R_1 + B_1)^2 \end{aligned}$$

where

$$(5.24) \quad \begin{aligned} A_1 &= \frac{1}{2} \\ B_1 &= \frac{1}{2}[p_1 \pi_1 - \rho_1 + \frac{b}{a}(\pi_1 - 1)] \\ D_1 &= \frac{1}{m_1^c} \end{aligned}$$

The $t+1$ -period problem by induction.

Assume

$$(5.25) \quad f_t(R_t) = (-1)^t D_t u(A_t R_t + B_t)^{t+1}, \quad u(\cdot) < 0$$

where

$$(5.26) \quad a(A_t R_t + B_t) + b > 0$$

and

$$D_t > 0, \quad A_t = \frac{1}{t+1}$$

we will show that (5.25) holds for $(t+1)$ and that the only condition required is

$$(5.27) \quad a(A_{t+1} R_{t+1} + B_{t+1}) + b > 0$$

The latter will be established when we show that (5.27) and optimization at step $t+1$ imply (5.26).

We substitute (5.25) in (5.8) and optimizing in the market for a fixed C_{t+1} we obtain: (recall (2.8))

$$(5.28) \quad \frac{d}{dR_t} f_t(R_t) \phi_{t+1}(\underline{x}_{t+1}) = \lambda P_{t+1}(\underline{x}_{t+1})$$

or under the condition that $\lambda > 0$

$$(5.29) \quad a(A_t R_t(\underline{x}_{t+1}) + B_t) + b = \left[(-1)^t \frac{\lambda a^t (c+1)^t}{D_t (t+1) A_t} \frac{P_{t+1}(\underline{x}_{t+1})}{\phi_{t+1}(\underline{x}_{t+1})} \right]^{\frac{1}{(t+1)(c+1)-1}}$$

Note that the quantity in brackets is positive if $\lambda > 0$ since $a(c+1) < 0$ by assumption.

From the budget constraint (5.6) and using (5.29), we obtain:

$$(5.30) \quad \left[\frac{\lambda a^t (c+1)^t (-1)^t}{D_t (t+1) A_t} \right]^{\frac{1}{(c+1)(t+1)-1}} = \frac{a A_t}{m_{t+1}} Q_{t+1}$$

where

$$(5.31) \quad \begin{aligned} Q_{t+1} &\equiv R_{t+1} - C_{t+1} + p_{t+1} \pi_{t+1} - \rho_{t+1} \\ &\quad + \frac{b}{a} \frac{\pi_{t+1}}{\Lambda_t} + \frac{B_t}{\Lambda_t} \pi_{t+1} \end{aligned}$$

and similarly to Chapter 3 m_{t+1} is the cost of a unit of risky asset: for period $t+1$,

$$(5.32) \quad m_{t+1} \equiv \int \left(\frac{P_{t+1}(\underline{x}_{t+1})}{\phi_{t+1}(\underline{x}_{t+1})} \right)^{\frac{1}{(t+1)(c+1)-1}} P_{t+1}(\underline{x}_{t+1}) d\underline{x}_{t+1}$$

where we must satisfy the condition:

$$(5.33) \quad \frac{A_t a}{m_{t+1}} Q_{t+1} > 0$$

because we require the left hand side of (5.30) to be > 0 since we demand $\lambda > 0$. Also, we observe that $m_{t+1} > 0$ since $P_{t+1}(\underline{x}) > 0$. Now if we denote $g = (t+1)(c+1)$, we observe that

$$(5.34) \quad E \left(\frac{P_{t+1}(\underline{\xi})}{\phi_{t+1}(\underline{\xi})} \right)^{g/g-1} = \int \left(\frac{P_{t+1}(\underline{x})}{\phi_{t+1}(\underline{x})} \right)^{\frac{1}{g-1}} P_{t+1}(\underline{x}) d\underline{x} \equiv m_{t+1}$$

Substituting (5.24), (5.29), (5.30) into (5.8) and using observation (5.34) the D.P. relation reduces to:

$$(5.35) \quad \begin{aligned} &f_{t+1}(R_{t+1}) = \\ &\max_{C_{t+1}} \left[-u(C_{t+1}) D_t m_{t+1} \frac{(-1)^t}{[a(c+1)]^{t+1}} \left(\frac{a A_t}{m_{t+1}} Q_{t+1} \right)^{(t+1)(c+1)} \right] \end{aligned}$$

Note that $(-1)^t [a(c+1)]^{t+1} < 0$, $D_t > 0$ by assumption, $m_{t+1} > 0$. Thus, we only need to maximize

$$(5.36) \quad \Delta \equiv u(C_{t+1}) \left(\frac{aA_t}{m_{t+1}} Q_{t+1} \right)^{(t+1)(c+1)}$$

with respect to C_{t+1} . [Recall $Q_{t+1} = Q_{t+1}(C_{t+1})$ by (5.31).]

By examining the second derivative of (5.36) and using similar algebra as in the 1-period problem, we see that Δ is strictly concave.⁺ Thus the first order condition guarantees a unique maximum as long as

$$(i) \quad aC_{t+1} + b > 0 \quad \text{and}$$

$$(ii) \quad \frac{A_t a}{m_{t+1}} Q_{t+1} > 0 \quad \text{which is the same as (5.33)}$$

The first order condition implies that

$$(5.39) \quad aC_{t+1}^* + b = \frac{a}{t+1} Q_{t+1}^*$$

or using the definition of Q_{t+1} , (5.31),

$$(5.40) \quad C_{t+1}^* = \frac{1}{t+2} R_{t+1} + \frac{1}{t+2} \left(p_{t+1} \pi_{t+1} - \rho_{t+1} + \frac{b}{a} \frac{\pi_{t+1}}{A_t} + \frac{B_t}{A_t} \tau_{t+1} - \frac{b}{a} (t+1) \right)$$

But because of (5.39) (i) and (ii) are equivalent, thus it is only required that

⁺ A detailed proof of the concavity of Δ is found in Appendix III.

$$(5.41) \quad aC_{t+1}^* + b > 0$$

Now let us summarize the conditions:

$$\begin{array}{ccc}
 (5.41) & \begin{array}{c} \rightleftharpoons \\ \uparrow \\ \text{because of} \\ \text{choice of } C_{t+1}^* \end{array} & (5.33) \quad \begin{array}{c} \rightleftharpoons \\ \uparrow \\ \text{because of the} \\ \text{budget constraint} \end{array} & \lambda > 0 \quad \begin{array}{c} \rightleftharpoons \\ \uparrow \\ \text{because of} \\ \text{choice of } R_t \end{array} a(A_t R_t + B_t) + b > 0 \\
 (5.42) \quad (5.39) & & (5.30) & (5.28)
 \end{array}$$

Finally, because of (5.39), (5.35) becomes,

$$f_{t+1}(R_{t+1}) = -u(C_{t+1}^*) m_{t+1} \frac{(-1)^t}{[a(c+1)]^{t+1}} D_t \left(\underbrace{\frac{aA_t}{m_{t+1}} Q_{t+1}^*}_{A_t \frac{t+1}{m_{t+1}} (aC_{t+1}^* + b) \text{ by (5.39)}} \right)^{(t+1)(c+1)}$$

Summary of Results for Model Ia₁:

$$\begin{aligned}
 f_{t+1}(R_{t+1}) &= (-1)^{t+1} D_{t+1} u(C_{t+1}^*)^{t+2} \\
 (5.43) \quad f_{t+1}(R_{t+1}) &= (-1)^{t+1} D_{t+1} u(A_{t+1} R_{t+1} + B_{t+1})^{t+2}
 \end{aligned}$$

with

$$(5.44) \quad C_{t+1}^* = A_{t+1} R_{t+1} + B_{t+1}$$

where

m_{t+1} was defined in (5.34) (the cost of a unit of risky asset)

where because of the form of the postreinsurance wealth (5.47)

a unit of risky asset was implicitly defined to be $\left(\frac{P_{t+1}(\xi_{t+1})}{\phi_{t+1}(\xi_{t+1})} \right)^{\frac{1}{(c+1)(t+1)-1}}$

Also

$$D_{t+1} = m_{t+1} \left(\frac{((t+1)A_t)}{m_{t+1}} \right)^{(t+1)(c+1)} D_t$$

$$(5.45a) \quad = \frac{1}{m_{t+1}^{(t+1)(c+1)-1}} D_t$$

since $A_t = \frac{1}{t+1}$ by the induction assumption

$$(5.45b) \quad A_{t+1} = \frac{1}{t+2}$$

$$(5.45c) \quad B_{t+1} = \frac{1}{t+2} \left(p_{t+1} \pi_{t+1} - \rho_{t+1} + \frac{b}{a} \frac{\pi_{t+1}}{A_t} + \frac{B_t}{A_t} \pi_{t+1} - \frac{b}{aA_t} \right)$$

with $D_0 = 1$, $B_0 = 0$, $A_0 = 1$ along with the condition (5.41):

$$(5.46) \quad a(A_{t+1}R_{t+1} + B_{t+1}) + b > 0$$

Let us check now that no other condition arises for maximization in step t . That is, we will show that

$$a(A_{t+1}R_{t+1} + B_{t+1}) + b > 0 \implies a(A_t R_t + B_t) + b > 0$$

This follows by observing that (5.46) is equivalent to (5.41) and then by checking the summary of conditions (5.42). Finally, from (5.28) and (5.30) we find that the optimal reinsurance treaty will transform the wealth of the I.C. to

$$R_t(\underline{x}) = \frac{Q_{t+1}^*}{m_{t+1}} \left(\frac{p_{t+1}(\underline{x}_{t+1})}{\phi_{t+1}(\underline{x}_{t+1})} \right)^{\frac{1}{(c+1)(t+1)-1}}$$

$$(5.47) \quad - \frac{b}{aA_t} - \frac{B_t}{A_t}$$

where, using the definition of Q_{t+1}^* along with (5.45), we find that

$$Q_{t+1}^* = (1 - A_{t+1}) \left(R_{t+1} + p_{t+1} \pi_{t+1} - \rho_{t+1} + \frac{b}{aA_t} \pi_{t+1} + \frac{B_t}{A_t} \pi_{t+1} \right) - \frac{b}{a} \frac{A_{t+1}}{A_t}$$

(5.48)

or since

$$(1 - A_{t+1}) = 1 - \frac{1}{t+2} = \frac{t+1}{t+2} = \frac{A_{t+1}}{A_t}$$

$$Q_{t+1}^* = (1 - A_{t+1}) \left(R_{t+1} + p_{t+1} \pi_{t+1} - \rho_{t+1} + \frac{b}{a} \left(\frac{\pi_{t+1}}{A_t} - 1 \right) + \frac{B_t}{A_t} \pi_{t+1} \right)$$

(5.49)

Further the solution is unique because of strict concavity. Section 5.4 compares these results to those of Chapter 3.

5.3.2 Model Ia₂

In this case we use the version of the Dynamic Programming relation given by (5.5) and proceed in exactly the same way as in Model Ia₁:

We fix C_1 and maximize in the market to obtain

$$R_0(\underline{x}) = \frac{\lambda^{1/c}}{a} \left(\frac{p_1(\underline{x}_1)}{\phi_1(\underline{x}_1)} \right)^{1/c} - \frac{b}{a}$$

(5.50)

and

$$\lambda^{1/c} = \frac{a}{m_1} \underbrace{\left(R_1 - C_1 + p_1 \pi_1 - \rho_1 + \frac{b}{a} \pi_1 \right)}_{Q_1}$$

(5.51)

along with the condition

$$(5.52) \quad \frac{aQ_1}{m_1} > 0$$

Substituting (5.50), (5.51) in (5.5) we see that we only have to maximize

$$(5.53) \quad u(C_1) \left(\frac{a}{m_1} Q_1 \right)^{c+1}$$

over C_1 .

The first and second derivatives are given once again by Δ_1 (5.17), Δ_2 (5.18) but now $a(c+1) > 0$ and $ac < 0$ which makes each term in Δ_2 (5.18) negative. Thus concavity is again assured. The rest of the arguments are the same as in Model Ia₁; the only difference lies in proving the strict concavity of

$$(5.54) \quad \Delta = u(C_{t+1}) \left(\frac{aA_t}{m_{t+1}} Q_{t+1} \right)^{(t+1)(c+1)}$$

but this is dealt with in Appendix III.

Summary of Results for Model Ia₂

$$(5.55) \quad f_{t+1}(R_{t+1}) = D_{t+1} u(A_{t+1} R_{t+1} + B_{t+1})^{t+2}$$

$$(5.56) \quad C_{t+1}^* = A_{t+1} R_{t+1} + B_{t+1}$$

where m_{t+1} , as defined in (5.34), is the cost of a unit of risky asset and D_{t+1} , A_{t+1} , B_{t+1} , $R_t(x)$ given by the same formulae as

in Model Ia₁: (5.45), (5.47), (5.48), (5.49), along with the condition (5.46) that the starting capital R_{t+1} must satisfy.

However, an extra condition is required now on t so that the $t+1$ -period problem has a solution as described above,

$$(5.57) \quad (t+1)(c+1) - 1 < 0$$

This condition is needed to keep $f_t(\cdot)$ concave, as it is required for maximization in the market. (If $(t+1)(c+1) > 1$ then $f_t(\cdot)$ is convex.) It is also needed to show the strict concavity of Δ .

Further, the solution is unique because of strict concavity.

5.4 Interpretation

The results are very similar to those of Chapter 3 for Models Ia₁, Ia₂ and thus we will be rather brief here. Sections 4.2 (Data Requirements) and 4.3 (The Optimal Dividend Payments Policies) apply here as well, but now the dividends do not depend on the cost m_t of the risky asset. Section 4.4 (The Optimal Reinsurance Treaty) is the same as far as the interpretation of the transactions among I.C., s/p-holders and the Market is concerned. Further, since $\frac{P_{t+1}}{\phi_{t+1}}$ is increasing in each x^i , $R_t(\underline{x})$ decreases in each x^i as long as $(t+1)(c+1) - 1 < 0$ for Model Ia₂, while it holds with no further conditions for Model Ia₁ (Figure 2a is again applicable). Section 4.5 is virtually the same, except that now the inequalities (4.7) become:

$$(5.58) \quad (1) \quad \text{Ia}_1, \quad (a > 0, c < -1) \quad \eta_{t+1}^g < m_{t+1} \leq \frac{g}{g-1}$$

$$(5.59) \quad (ii) \quad Ia_2, \quad (a > 0, -1 < c < 0) \quad \eta_{t+1}^g > m_{t+1} \geq \pi_{t+1}^{\frac{g}{g-1}}$$

where

$$\eta_{t+1} \equiv E \left(\frac{p_{t+1}(\xi_{t+1})}{\phi_{t+1}(\xi_{t+1})} \right)^{\frac{1}{g-1}}$$

is the expected value of a unit of risky asset, $g \equiv (t+1)(c+1)$ and m_{t+1} , as defined in (5.32), is the cost of a unit of risky asset.

For the multiplicative utility case we cannot extend our results to an infinite horizon. This is obvious for Model Ia_2 because of the time condition (5.57). For Model Ia_1 C_t^* converges to $-\frac{b}{a}$ as $t \rightarrow \infty$, while the reinsurance formula (5.47) reduces to $R_{\text{new}} = \left(R_{\text{old}} + p\pi - \rho - \frac{b}{a} \right) \frac{1}{\pi}$ which implies that the I.C. will get rid of all its risky portfolio (i.e. it will become an intermediary agent). These limiting policies are hardly acceptable on two accounts: (1) The utility functions of class Ia_1 tend to $-\infty$ when the dividend tends to $-\frac{b}{a}$. (2) The reserves process (the reinsurance transformation of wealth) becomes deterministic and the I.C. cannot tolerate the idea of risk-bearing; a situation that contradicts the name "Insurance Company". This type of result was expected because of the temporal risk aversion exhibited by the multiplicative form of Model Ia_1 which increases linearly with time (see Section 5.5/2 and formula (5.61) below).

The arguments regarding the right to liquidate the I.C. (4.7.2) are easily adaptable to the present situation. Finally, the decision for expenditures to promote sales (Section 4.7.3) holds in exactly

the same form except that now we use the formulae of Chapter 5 regarding B_{t+1} , A_{t+1} , D_{t+1} . Specifically, corresponding to (4.21), we will now have:

$$(5.60) \quad B_t = A_t \left[sv_t - r_t + \frac{B_{t-1}}{A_{t-1}} \pi + \frac{b\pi}{aA_{t-1}} - \frac{b}{aA_{t-1}} \right]$$

where s, v_t, r_t as defined in Section 4.7.3.

5.5 Remarks

1) Only Models Ia₁, Ia₂ give us closed form results in the multiplicative utility case. Let's see why this does not happen with Models Ia₃, Ib, II.

Ia₃: The problem here lies with the second order condition Δ_2 . Namely, solving for C_{t+1}^* in $\Delta_1 = 0$ and substituting in $\Delta_2 < 0$, we see that it requires that $a > 0$ which is contradictory to Ia₃.

Ib: The first problem here is that $\frac{1}{a} \log(ax + b)$, $a > 0$, $ax + b > 0$ is partly positive and partly negative. We can overcome this by restricting the region of x to either $ax + b > 1$ or $ax + b < 1$. Still, we cannot arrive at a simple solution even for the one-period problem.

II: This case is uninteresting by its very nature. It seeks

to maximize $-\prod_{i=1}^N |u(C_i)|$ where $u(C_i) = -\frac{1}{\gamma} e^{-\gamma C_i}$, $\gamma > 0$. But

this is equivalent to max. $\sum_{i=1}^N C_i$ and there is no impatience or

discounting. However, since the interest rate is positive ($\pi_t < 1$), it will require that we consume (pay as dividend) *nothing* and even

have C_t as negative as possible until the end of the program when everything is given as dividends.

2) Models Ia_1 , Ia_2 are quite different in nature from the point of view of risk aversion:

$$(5.61) \quad -\frac{f''_t(R_t)}{f'_t(R_t)} = -\frac{u''(A_t R_t + B_t)}{u'(A_t R_t + B_t)} - t \frac{u'(A_t R_t + B_t)}{u(A_t R_t + B_t)}$$

The first term is the usual risk aversion index. The second term is called by Meyer (1969) the *temporal risk-aversion*. For Model Ia_1 ($u(\cdot) < 0$) the I.C. is *temporally risk-averse* as the second term increases with time. The opposite is true for Model Ia_2 ($u(\cdot) > 0$). In this case the I.C. is *temporally risk-seeking*. In the case of additive utility, the second term above does not enter the expression and the I.C. is *temporally risk neutral*.

The meaning of temporal risk-aversion is the following: A temporal risk averse individual would be more risk averse in gambles that determine his stream of consumption over a longer period of time than for those for a shorter period of time. The opposite holds for a temporally risk-seeking individual. In this respect Model Ia_1 seems to have more meaning. However, we cannot preclude the possibility of cases where Model Ia_2 applies as long as the number of periods is limited so that $(t+1)(c+1) - 1 < 0$ as required by (5.57).

Finally, we note that the Reinsurance Market does not depend on how each I.C. aggregates each utility function over time. Therefore, the choice between the additive or multiplicative forms is value judgement of the management of each I.C.

CHAPTER 6

CONCLUSIONS

One of the major criticisms of the models presented in this thesis is the data requirements, which in practice must be calculated using forecasting techniques. In future work it would be of interest therefore to examine if simple forms can still be found when the quantities $P_t(\cdot)$, $\phi_t(\cdot)$, p_t are directly treated as random and are updated possibly by Bayesian techniques. It is possible that credibility theories [see for example Jewell (1973a), (1973b), (1974); Pechlivanides (1973)] might provide the answer to this question.

The other major criticism is of course the Reinsurance Market, which in reality does not exist as described. It might be interesting to study how such a market mechanism can be implemented. The key issue is the determination of the equilibrium price function. This is basically a question in the field of economics and could be achieved, for instance, by the establishment of a central bureau that proposes a price function and then reviews it based on a feedback process with the participating I.C.'s. Another approach might be to actually create "elementary contracts" as described in Chapter 2, which are traded between participants like securities in a stock market.

The Reinsurance Market we described allows only contracts that terminate within the same period. That is, all contracts formed at the start of a period must depend on the outcomes of the claims of that period only. At the end of the period, when the values of the claims random variables are known, all debts are paid, the

contracts are fulfilled, and we start anew the next period. The question, therefore, is: Can we extend the Reinsurance Market so as to handle contracts extending for more than one period? Of course, this question is possibly of theoretical rather than practical interest.

The concept of a price function was already proposed as a method for calculating premiums in usual insurance (see Section 4.7.1 and Appendix II). The premiums calculated with the price function are no more than an expected value w.r.t. to probability density function $\frac{P(x)}{\pi}$. Credibility theories in premium calculation should directly apply to this new type of "expected value." It is then of interest to see how the form of the variable loading factor $f(x)$ (introduced in Appendix II) couples with the credibility formulae.

In Section 4.7.3 we generalized the models to include a decision to promote sales. It was assumed that an "average" type of policy was sold to the customers and thus the whole portfolio of the I.C. (premiums-claims) was $v\zeta$ where v was the "volume of policies sold." It might be interesting to examine the same problem when the portfolio of premiums-claims is given by $v_1\zeta_1 + v_2\zeta_2 + \dots + v_m\zeta_m$ i.e., it is a mixture of varying quantities v_i of different types of risks ζ_i . In this way the model could be further generalized to include underwriting improvement when the appropriate cost functions are introduced.

APPENDIX I

Lemma 2:

Let $\underline{k}^0 \in \partial K$, as $\underline{k} \rightarrow \underline{k}^0$ $\lim \bar{d}(\underline{k})$ and $\lim w(\underline{k})$ either both exist and are strictly positive and all the $\bar{d}_j(\underline{k})$ have finite limits or both are $+\infty$.

Proof:

For convenience, we denote $\lim_{\underline{k} \rightarrow \underline{k}^0}$ simply \lim . There are two cases:

Case 1. $\Pr[\underline{\psi} \in Y_\infty] = 0$

$0 < L(\underline{y}, \underline{k}^0) < \infty$ with probability 1.

Let $J(\underline{k}^0) \equiv \{j \in N \mid k_j^0 = 0, \underline{k}^0 \in \partial K\}$; $J(\underline{k}^0) \neq \emptyset$ by definition of ∂K . Also $N - J(\underline{k}^0)$ is nonempty since $(0, \dots, 0) \notin K$.

(a) $j \in J(\underline{k}^0)$:

$$\lim k_j \frac{P}{\phi} = (\lim k_j) L(\underline{y}, \underline{k}^0) = 0 \text{ on } \underline{y} \in Y_0 \quad (Y_0 \equiv Y - Y_\infty)$$

$$u_j^{-1}\left(k_j \frac{P}{\phi}\right) \rightarrow \bar{t}_j > \bar{y}_j \geq y_j \text{ by A.1 and A.2}$$

Thus

$$\int_{Y_0} \left[\lim u_j^{-1}\left(k_j \frac{P}{\phi}\right) - y_j \right] L(\underline{y}, \underline{k}^0) \phi(\underline{y}) d\underline{y} > 0 \text{ (possibly } +\infty \text{ if } \bar{t}_j \text{ is } +\infty)$$

But the above integral is equal to the integral over Y (since $\Pr[\underline{\psi} \in Y_\infty] = 0$). Thus by uniform convergence and the definition of $d_j(\underline{k})$ it follows that $\lim d_j(\underline{k})$ is positive (possibly $+\infty$) \Rightarrow $\lim \bar{d}_j(\underline{k}) > 0$ (possibly $+\infty$).

(b) $j \in N - J(\underline{k}^0)$: Then $k_j^0 > 0$ and $u_j^{-1}\left(k_j \frac{p}{\phi}\right) + u_j^{-1}\left(k_j^0 L\right)$
and the limit of $d_j(\underline{k})$ is defined (possibly $+\infty$ or $-\infty$). Hence,
 $\lim \bar{d}_j(\underline{k})$ exists and is nonnegative or $+\infty$.

(c) Thus $\lim \bar{d}(\underline{k})$ exists and is positive or $+\infty$.

(d) Since, in a neighborhood of \underline{k}^0 $d_j(\underline{k}) > 0$, $j \in J$ by (a),

and in order to satisfy $\sum_{i=1}^n d_i(\underline{k}) = 0$, for each \underline{k} in this neighborhood, there must exist a $h \in N - J(\underline{k}^0)$, $h = h(\underline{k})$, with
 $d_h(\underline{k}) < 0 \Rightarrow \bar{d}_h(\underline{k}) = 0$

then

$$w_k(\underline{k}) = |\bar{d}(\underline{k})(k_h - k_\ell)| = \bar{d}(\underline{k}) |k_h - k_\ell|$$

Now $k_\ell \rightarrow 0$ since $k_\ell \equiv \min k_i$ thus

$$\lim w_h(\underline{k}) = (\lim k_h) \lim \bar{d}(\underline{k})$$

but $\lim k_h > 0$ since $h \notin J(\underline{k}^0)$.

Thus if $\bar{d}(\underline{k}) \rightarrow +\infty \Leftrightarrow w_h(\underline{k}) \rightarrow +\infty \Rightarrow w(\underline{k}) \rightarrow +\infty$.

(e) Summary:

We have the following possibilities.

- (1) All $\bar{d}_j(\underline{k})$ have finite nonnegative limits with at least one being strictly positive since $J(\underline{k}^0)$ is nonempty
 $\Rightarrow \bar{d}(\underline{k})$ has a finite positive limit (f.p.l.)
 \Rightarrow each $w_j(\underline{k})$ has a f.p.l. $\Rightarrow w(\underline{k})$ has a f.p.l.
- (2) Some $\bar{d}_j(\underline{k})$ becomes $+\infty \Rightarrow \bar{d}(\underline{k}) \rightarrow +\infty \Rightarrow w(\underline{k}) \rightarrow +\infty$.

Case 2. $\Pr[\underline{\psi} \in Y_\infty] > 0$

$$0 < L(\underline{y}, \underline{k}^0) < +\infty, \underline{y} \in Y_0$$

$$L(\underline{y}, \underline{k}^0) = +\infty, \underline{y} \in Y_\infty$$

Let $j \in N - J(\underline{k}^0)$ which is nonempty. Then since $k_j^0 > 0$, $k_j > 0$ in a neighborhood of \underline{k}^0 .

$$\text{Let } z_j(\underline{y}, \underline{k}) = u_j^{-1} \left(k_j \frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})} \right)$$

From the definition of $d_j(\underline{k})$ we can write

$$d_j(\underline{k}) = \int_{Y_\infty} [z_j(\underline{y}, \underline{k}) - y_j] \frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})} \phi(\underline{y}) d\underline{y} + \int_{Y_0} [z_j(\underline{y}, \underline{k}) - y_j] \frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})} \phi(\underline{y}) d\underline{y}$$

The second integral: We said that $k_j > 0$ in a neighborhood of \underline{k}^0 . Also since $\frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})}$ cannot have zero as a limit point for any $\underline{y} \in Y$ and $\underline{k} \in K$ (recall remark 2 of Assumption A.3) it follows that $k_j \frac{P}{\phi} + k_j^0 L > 0$. But then $\lim z_j(\underline{y}, \underline{k}) < \bar{t}_j$. Also y_j is always finite (by A.2) thus $z_j(\underline{y}, \underline{k}) - y_j$ is bounded for all \underline{y} in a neighborhood of \underline{k}^0 . Finally

$$\int_{Y_0} \lim [z_j(\underline{y}, \underline{k}) - y_j] L(\underline{y}, \underline{k}^0) \phi(\underline{y}) d\underline{y} = \lim \int_{Y_0} [z_j(\underline{y}, \underline{k}) - y_j] \frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})} d\underline{y}$$

because of the uniform convergence assumption. Further since $z_j - y_j < +\infty$ (proved above) and $L(\underline{y}, \underline{k}^0) < +\infty \forall \underline{y} \in Y_0$ by assumption, the R.H.S. is $< +\infty$ (since each y_j varies over a finite interval by A.2).

The first integral: Over Y_∞ $k_j^0 L(\underline{y}, \underline{k}^0) = +\infty \Rightarrow z_j(\underline{y}, \underline{k}) \rightarrow \underline{t}_j$. But $\underline{t}_j < \underline{y}_j \leq y_j$ for all $\underline{y} \in Y$ thus $z_j(\underline{y}, \underline{k}) - y_j < 0$ for all $\underline{y} \in Y_\infty$ in a neighborhood of \underline{k}^0 . Also since $\frac{P(\underline{y}, \underline{k})}{\phi(\underline{y})} \rightarrow +\infty$ for all $\underline{y} \in Y_\infty$ uniformly the first integral diverges to $-\infty$ as $\underline{k} \rightarrow \underline{k}^0$.

Combining our observations for the first and second integral we conclude that $d_j(\underline{k}) \rightarrow -\infty$ as $\underline{k} \rightarrow \underline{k}^0$ for $j \in N - J(\underline{k}^0)$.

However, we know that $\sum_{j \in N} d_j(\underline{k}) = 0$ (by the conservation condition (2.12) or (2.16)). Thus if one $d_j(\underline{k})$ tends to $-\infty$, the sum of those $d_j(\underline{k})$'s that are positive must tend to $+\infty \Rightarrow \bar{d}(\underline{k}) \rightarrow +\infty$.

Now the argument of Case 1 applies that for each \underline{k} in a neighborhood of \underline{k}^0 there is at least one $h \in N - J(\underline{k}^0)$ such that $d_h(\underline{k}) < 0$ (in fact any $j \in N - J(\underline{k}^0)$ will do the job since $d_j(\underline{k}) \rightarrow -\infty$ as we showed above).

$$\Rightarrow \bar{d}_h(\underline{k}) = 0 \Rightarrow w_h(\underline{k}) = \bar{d}(\underline{k}) |k_h - k_{\ell}(\underline{k})|$$

Since $k_{\ell}(\underline{k}) \rightarrow 0$, $\bar{d}(\underline{k}) \rightarrow +\infty$, and $k_h^0 > 0$ it follows that $w_h(\underline{k}) \rightarrow +\infty \Rightarrow w(\underline{k}) \rightarrow +\infty$.

To conclude this case therefore

$$\bar{d}(\underline{k}) \rightarrow +\infty \text{ and } w(\underline{k}) \rightarrow +\infty.$$

Combining now the conclusions of Case 1 and Case 2, we have the statement of the Lemma.

Theorem 2:

If A1, A2, A3 hold, Problem II has a solution.

Proof:

We first state

Brouwer's Fixed Point Theorem:

Let K be a compact and convex subset of \mathbb{R}^n and f a continuous mapping from K into K . Then there exists a fixed point

\hat{x} for f , i.e. $\hat{x} = f(\hat{x})$. (For a proof see Berge (1959).)

Now in our problem:

- (i) K , as defined in (2.15), is a convex and compact set in \mathbb{R}^n .
- (ii) Define a continuous function $\alpha(x)$ on $0 \leq x \leq +\infty$,
 $0 \leq \alpha(x) \leq 1$, $\alpha(0) = 0$, $\alpha(x) > 0$ for $x > 0$,
 $\alpha(x) = 1$ for $x \geq D$ where $D > 0$.

We are going to look at $\alpha(w(\underline{k}))$ and denote it $\alpha(\underline{k})$.[†]

Define

$$(I.1) \quad \bar{d}_j(\underline{k}) \equiv \begin{cases} (1 - \alpha(\underline{k}))\bar{d}_j(\underline{k}) + \alpha(\underline{k}) & (j = 1, \dots, n) \\ 1 & \text{if } w(\underline{k}) \text{ is undefined (i.e. } +\infty). \end{cases}$$

Now $\bar{d}_j(\underline{k})$ is continuous for $\underline{k} \in K$. This is obvious for \underline{k} such that $w(\underline{k})$ is defined ($< +\infty$) (note that by Lemma 2, we cannot have $\bar{d}(\underline{k})$, $w(\underline{k})$ well defined ($< +\infty$) while some $\bar{d}_j(\underline{k})$ are undefined or $+\infty$). For points \underline{k}^0 that $w(\underline{k}^0) = +\infty$ (necessarily $\underline{k}^0 \in \partial K$), we defined $\bar{d}_j(\underline{k}^0) = 1$. Then we can find a neighborhood of \underline{k}^0 so that $w(\underline{k})$ is defined and $w(\underline{k}) > D$. Thus in this neighborhood, $\alpha = 1$ and $\bar{d}_j(\underline{k}) = 1$. Therefore, $\bar{d}_j(\underline{k})$ is continuous on $\underline{k} \in K$.

Finally we define the mapping

$$(I.2) \quad \tilde{k}_j = \frac{k_j + \bar{d}_j(\underline{k})}{1 + \sum_{i=1}^n \bar{d}_i(\underline{k})} \quad (j = 1, \dots, n)$$

[†]This method is borrowed from Arrow-Hahn (1971), pp. 31-32.

since $\bar{d}_1(\underline{k}) \geq 0$ for all \underline{k} the mapping is well defined. It is

easy to check that $\tilde{k}_j \geq 0$ and $\sum_{j=1}^n \tilde{k}_j = 1$

(iii) thus the mapping (I.2) is from K into K

(iv) the mapping (I.2) is continuous for $\underline{k} \in K$. We already

showed that $\bar{d}_j(\underline{k})$ is continuous for all j

(v) It follows from the fixed point theorem that there exists

\underline{k}^* so that

$$(I.2)^* \quad k_j^* = \frac{k_j^* + \bar{d}_j(k^*)}{1 + \sum_{i=1}^n \bar{d}_i(k^*)} \quad j = 1, \dots, n.$$

We denote for simplicity

$\bar{d}_1(k^*), \bar{d}_1(k^*), \bar{d}(k^*), \alpha(k^*)$ by $\bar{d}_1^*, \bar{d}_1^*, \bar{d}^*, \alpha^*$ respectively.

(vi) $k_j^* \neq 0$ for all j .

Suppose to the contrary that $k_j^* = 0$ then by (I.2)*

$$(1 - \alpha^*) \bar{d}_j^* + \alpha^* = 0 \Rightarrow \alpha^* = 0, \bar{d}_j^* = 0$$

but by definition of α , this implies $w(\underline{k}^*) = 0$. But $\underline{k}^* \in \partial K$

and we proved in Lemma 2, Chapter 2 that for $\underline{k} \in \partial K$ $w(\underline{k}) > 0$ (possibly $+\infty$) and we reach a contradiction. Thus, $k_j^* > 0$ for all j .

(vii) Since \underline{k}^* belongs to the interior of K , it follows that $\bar{d}^* < \infty$. We consider (I.2)* and drop the use of asterisk after substituting for \bar{d}_j

$$(I.3) \quad k_j = \frac{k_j + (1 - \alpha) \bar{d}_j + \alpha}{1 + (1 - \alpha) \bar{d} + n\alpha}$$

or by multiplying both sides by the denominator of the right hand side and rearranging:

$$(I.4) \quad \bar{d}k_j - \bar{d}_j = \alpha(1 + \bar{d}k_j - nk_j - \bar{d}_j) .$$

Now since $\sum d_j(\underline{k}) = 0$ for $\underline{k} \in \text{interior } (i')$, it follows that there is an h so that $d_h(\underline{k}) \leq 0 \Rightarrow \bar{d}_h(\underline{k}) = 0$. In this case, (I.4) becomes

$$(I.5) \quad \bar{d}k_h = \alpha(1 + \bar{d}k_h - nk_h) .$$

Now let J_1 be the set of indices for which

$$1 + \bar{d}k_j - nk_j - \bar{d}_j = 0$$

and $J_2 = N - J_1$ (the rest of the indices). We note that because of (I.4)

$$j \in J_1 \Rightarrow \bar{d}k_j - \bar{d}_j = 0 \Rightarrow k_j = \frac{1}{n} .$$

It follows that if $k_j \neq \frac{1}{n} \Rightarrow j \in J_2$.

We have the following cases

I. $J_2 = \emptyset$

Then $J_1 = N \Rightarrow k_j = \frac{1}{n}$ for all $j \in N \Rightarrow w(\underline{k}) = 0 \Rightarrow \alpha = 0$.

Thus (I.3) becomes

$$k_j = \frac{k_j + \bar{d}_j}{1 + \bar{d}}$$

and for an h $\bar{d}_h = 0 \Rightarrow 1 + \bar{d} = 1 \Rightarrow \bar{d} = 0$.

II. $J_2 \neq \emptyset$, $k_h = \frac{1}{n}$

For $k_h = \frac{1}{n}$ (I.5) becomes

$$\bar{d}k_h = \alpha \bar{d}k_h.$$

If $\bar{d} \neq 0 \Rightarrow \alpha = 1$. Then (I.3) becomes

$$k_j = \frac{k_j + 1}{1+n} \Rightarrow k_j = \frac{1}{n} \text{ for all } j$$

but then $w(k) = 0 \Rightarrow \alpha = 0$ contradiction. Thus $\bar{d} = 0$.

III. $J_2 \neq \emptyset$, $k_h \neq \frac{1}{n}$

For a $j \in J_2$ solve (I.4) for α . Do the same with (I.5) and equate

$$(I.6) \quad \alpha = \frac{k_j \bar{d} - \bar{d}_j}{1 + k_j \bar{d} - k_j n - \bar{d}_j} = \frac{k_h \bar{d}}{1 + k_h \bar{d} - k_h n}$$

and since $0 \leq \alpha \leq 1 \Rightarrow 1 - k_h n \geq 0 \Rightarrow k_h \leq \frac{1}{n}$ but $k_h \neq \frac{1}{n} \Rightarrow k_h < \frac{1}{n}$.

Now multiplying diagonally and simplifying, we have:

$$(I.7) \quad \bar{d}(k_j - k_h) = \bar{d}_j(1 - k_h n) \quad j \in J_2$$

while for $j \in J_1$, $\bar{d}k_j - \bar{d}_j = 0$ and $k_j = \frac{1}{n}$. Thus if $\bar{d} \neq 0 \Rightarrow k_j = \frac{1}{n} = \frac{\bar{d}_j}{\bar{d}}$ which we note satisfies (I.7) too. Thus (I.7) holds for all $j \in N$ if $\bar{d} \neq 0$.

Now $k_h = \min_{i \in N} (k_i)$. For $j \in J_1$, $k_h < k_j = \frac{1}{n}$. Now suppose

to the contrary that there exists $j \in J_2$ such that $k_j < k_h$. Then

(I.7) implies that $\frac{\bar{d}_j}{\bar{d}} < 0$ (which is a contradiction) since

$$1 - k_h n > 0.$$

Therefore (I.7) holds for all j and $k_h = \min_1(k_1)$, but this implies $w(h) = 0$ (check def. of $w(k)$) $\Rightarrow \alpha = 0 \Rightarrow \bar{d} = 0$ with the same argument as in Case I: this is a contradiction, since we assumed $\bar{d} \neq 0$. Thus $\bar{d} = 0$.

Conclusion: In all three cases, $\bar{d} = 0 \Rightarrow d_j(k^*) = 0$ for all j . Thus Problem II has a solution with $k_1^* > 0$. ||

APPENDIX II

A PREMIUM CALCULATION PRINCIPLE

(a) The I.C. wants to calculate a premium for assuming the risk ξ . For this purpose it constructs a price function $P_{\xi}(x)$ where $P_{\xi}(x)dx$ will have the meaning of the price the I.C. asks in order to pay \$1 if $\xi = x$. Thus for assuming a risk ξ , the I.C. must calculate as premium

$$(1) \quad P[\xi] = \int x P_{\xi}(x) dx.$$

But it is conceivable that the customer may not wish to cover the whole risk ξ . Instead he might wish to participate in the risk by making a contract $Z(x)$ that pays to him $\$Z(x)$ when $\xi = x$. The premium accordingly should be,

$$(2) \quad P[Z(\xi)] = \int Z(x) P_{\xi}(x) dx.$$

Suppose now $\psi \equiv Z(\xi)$. As ψ is itself a random variable (r.v.) it is associated with a price function $P_{\psi}(y)$. Then the premium for assuming ψ should be

$$(3) \quad P[\psi] = \int y P_{\psi}(y) dy$$

but if the I.C. is consistent, (3) must be the same with (2). This implies that

$$(4) \quad P_{\xi}(x) = P_{\psi}(Z(x)) | Z'(x) |.$$

(b) A way to think of $P_\psi(y)$ is as a distortion of the density function $\phi_\psi(y)$ of the random variable ψ :

$$(5) \quad P_\psi(y) = f_\psi(y)\phi_\psi(y).$$

For example, $f_\psi(y) = a+by$. This would mean that in calculating the premium for ψ we are interested in the first and second moments only. If, however, $\psi = \xi^2 = Z(\xi)$ then (3) requires that

$$P_\xi(x) = f_\psi(x^2)\phi_\psi(x^2) |2x|$$

$$\text{but } |2x| \phi_\psi(x^2) = \phi_\xi(x)$$

$$P_\xi(x) = f_\psi(x^2)\phi_\xi(x) \text{ but by definition of } P_\xi(x), (5),$$

$$f_\psi(x^2) = f_\xi(x) = a+bx^2.$$

The above imply that whereas the premium for ψ is based on the first and second moments of ψ , the premiums for ξ is based on the first and third moments of ξ .

It can then be argued that this principle for premium calculation uses different criteria for different distributions, whereas the classical principles would use the same characteristic for all distributions (for instance, the second moment). This is exactly the point. It is doubtful whether we would like to look at the second moment for all distributions. It is rather more likely that we have to adopt our criterion to each specific distribution.

This principle of premium calculation is also *additive*. In effect, it is a way of extending the expected value principle by distorting the underlying distribution of the risk. In the example above we used $f_x(y) = a+by$ but

$$(6) \quad f_{\psi}(Z(x)) = f_{\xi}(x)$$

holds in general because of (4).

Also note that in particular, if $f_{\psi}(y) = \text{constant}$ for some r.v., ψ , then it must be constant for all other r.v.'s. This leads us to the particular case of calculating the premium based on the expected value only. Thus, the expected value principle is a particular case of the principle presented here.

(c) We showed above how from one price function we could determine another for a different random variable. A more systematic way of doing this is to start with $f_{\psi}(y)$ (assumed to be determined by the I.C. somehow) when ψ is uniformly distributed on $[0,1]$. Suppose now we want to find $f_{\xi}(x)$ for a r.v. $\xi \sim F_{\xi}(x)$. Certainly

$$\psi = F_{\xi}(x) \quad \text{where } \psi \sim U[0,1]$$

meaning that the r.v. $F_{\xi}(\xi)$ is uniformly distributed on $[0,1]$.

In particular, $y = F_{\xi}(x)$. Then because of (6)

$$(7) \quad f_{\psi}(F_{\xi}(x)) = f_x(x)$$

and $f_x(x)$ is determined as was desired.

APPENDIX III

Consider the conditions

$$(i) \quad aC_{t+1} + b > 0$$

$$(ii) \quad \frac{aQ_{t+1}}{m_{t+1}} > 0$$

$$(iii) \quad (t+1)(c+1) - 1 < 0.$$

We will show that Δ given by (5.36) or (5.54) is strictly concave for model Ia₁ if conditions (i) and (ii) hold and for model Ia₂ if conditions (i), (ii), (iii) hold.

Proof:

For both models,

$$\Delta = u(C_{t+1}) \left(\frac{aA_t}{m_{t+1}} Q_{t+1} \right)^{(t+1)(c+1)} \quad (5.36) \text{ or } (5.54)$$

The first derivative w.r.t. C_{t+1} is

$$\Delta_1 = u'(C_{t+1}) \left(\frac{aA_t}{m_{t+1}} Q_{t+1} \right)^{(t+1)(c+1)} -$$

$$u(C_{t+1}) \left(\frac{aA_t}{m_{t+1}} Q_{t+1} \right)^{(t+1)(c+1)-1} \frac{(t+1)(c+1)aA_t}{m_{t+1}}.$$

The second derivative w.r.t. C_{t+1} is

$$\begin{aligned} \Delta_2 &= u''(C_{t+1}) \left(\frac{aA_t}{m_{t+1}} Q_{t+1} \right)^{(t+1)(c+1)} - \\ &2u'(C_{t+1}) \left(\frac{aA_t}{m_{t+1}} Q_{t+1} \right)^{(t+1)(c+1)-1} \frac{(t+1)(c+1)aA_t}{m_{t+1}} + \\ &u(C_{t+1}) \left(\frac{a}{m} A_t Q_{t+1} \right)^{(t+1)(c+1)-2} \frac{a^2}{m^2} A_t^2 (t+1)(c+1)[(t+1)(c+1)-1]. \end{aligned}$$

Since $u(x) = \frac{(ax+b)^{c+1}}{a(c+1)}$ for both models, Δ_2 becomes

$$\begin{aligned} \Delta_2 &= K \left\{ ac \left(\frac{a}{m} AQ \right)^2 - 2(aC_{t+1} + b) \left(\frac{a}{m} AQ \right) \frac{aA}{m} (t+1)(c+1) + \right. \\ (1) \quad &\left. (aC_{t+1} + b)^2 \frac{A^2}{m^2} a(t+1)[(t+1)(c+1)-1] \right\} \end{aligned}$$

where we dropped the subscripts in A_t, m_{t+1}, Q_{t+1} as there is no danger of confusion and we called

$$(2) \quad K \equiv (aC_{t+1} + b)^{c-1} \left(\frac{a}{m} AQ \right)^{(t+1)(c+1)-2}$$

(note that $K > 0$ by assumption (i), (ii)).

Thus, we only have to investigate the sign of $\frac{\Delta_2}{K} \equiv \Delta'_2$. We can now write Δ'_2 in another form:

$$\begin{aligned} \Delta'_2 &= a(c+1) \left[\frac{a}{m} AQ - (aC_{t+1} + b) \frac{A}{m} (t+1) \right]^2 - \\ (3) \quad &\frac{aA^2}{m^2} \left(a^2 Q^2 + (aC_{t+1} + b)^2 (t+1) \right) \end{aligned}$$

Model Ia₁: ($a > 0$, $c < -1$). Look at (3).

Since $a(c+1) < 0$ and the second term is strictly negative, it follows that $\Delta'_2 < 0$.

Model Ia₂: ($a > 0$, $-1 < c < 0$).

Here $a(c+1) > 0$ but $1 > c+1 > 0$ and also $(t+1)(c+1)-1 < 0$. Take (1) then each term is negative thus $\Delta'_2 < 0$.

APPENDIX IV

Consider the one period problem once C_1 is fixed

P1:

$$\max_{R(\cdot)} Eu(R(\underline{x}))$$

$$\text{s.t.} \quad \int R(\underline{x})P(\underline{x})d\underline{x} = \int \left(\frac{R_1 - C_1}{\pi} + P_1 - x_1 \right) P(\underline{x})d\underline{x} \equiv m \quad (\text{constant})$$

where $P(\underline{x})$, C_1 , $R(\underline{x})$, P_1 , π , x_1 defined as before. Assume that $P(\underline{x})$ and $R(\underline{x})$ can be written as

$$(1) \quad P(\underline{x}) = \sum_{i=1}^{\infty} p_i g_i(\underline{x})$$

$$(2) \quad R(\underline{x}) = \sum_{i=1}^{\infty} d_i g_i(\underline{x})$$

where $\{g_i(\underline{x})\}_{i=1}^{\infty}$ is a complete orthonormal set of functions with $g_1(\underline{x}) \equiv 1$. It follows that $p_1 = \pi$ since $p_i \equiv \int P(\underline{x})g_i(\underline{x})d\underline{x}$, $d_i \equiv \int R(\underline{x})g_i(\underline{x})d\underline{x}$. Further assume that $p_i \neq 0$ for all i and define

$$(3) \quad \beta_i \equiv \frac{g_i(\underline{x})}{p_i}, \quad \left(\beta_1 = \frac{1}{\pi} \equiv r \right)$$

for all i .

$$(4) \quad z_i \equiv d_i p_i$$

Note that β_i ($i \neq 1$) can take both positive and negative values.

The budget constraint then becomes:

$$(5) \quad \sum_{i=1}^{\infty} z_i = m$$

and our objective is to

$$(6) \quad \max_{\{Z_i\}_{i=1}^{\infty}} Eu \left(\sum_{i=1}^{\infty} Z_i \beta_i \right)$$

If we solve for Z_1 in (5) and substitute in (6) the problem P1 becomes,

$$P2: \quad \max Eu \left(\sum_{i=2}^{\infty} Z_i (\beta_i - r) + mr \right)$$

which apart from the summation being extended to ∞ and the β_i 's being unconstrained to the positive semi-axis is similar to Hakansson's problem. Hakansson's solvency constraint is mathematically needed to ensure that the argument of u remains within the domain where u is defined. Analogous conditions are found to be necessary in our approach (for example for Model Ia $aR + b > 0$ which here

translates to $\sum_{i=2}^{\infty} Z_i (\beta_i - r) + mr > 0$).

The no-easy money condition can also be shown to be satisfied.

Consider the condition

$$(7) \quad \Pr \left[R(\underline{\xi}) - \frac{1}{\pi} \int R(\underline{x}) P(\underline{x}) d\underline{x} < 0 \right] > 0$$

Substituting we see that (7) implies

$$(8) \quad \Pr \left[\sum Z_i (\beta_i - r) < 0 \right] > 0$$

which is the no-easy money condition. It only remains to show that (7) is satisfied. First recall that $0 < \frac{P(\underline{x})}{\phi(\underline{x})} < \infty$. It follows

that $\frac{P(\underline{x})}{\pi}$ is another probability density function which is positive whenever and only whenever $\phi(\underline{x})$ is positive. Then suppose that (7) did not hold. Then with probability 1 (w.r.t. any of the two density functions $\frac{P(\underline{x})}{\pi}$ or $\phi(\underline{x})$)

$$R(\underline{\xi}) \geq \int R(\underline{x}) \frac{P(\underline{x})}{\pi} d\underline{x}$$

or $R(\underline{\xi}) \geq ER(\underline{\xi})$ with probability 1 but this is impossible. Thus

the fact that $\frac{P(\underline{x})}{\pi}$ behaves like a density function and is positive whenever $\phi(\underline{x})$ is positive (and vice versa) guarantees that (7) holds.

Once the problem is in the form P2 we could continue by modifying Hakansson's proof of the existence of optimal $\{Z_i\}_{i=2}^{\infty}$.

This, however, would be an unnecessary effort because to find the optimal solution explicitly we would be obliged to once again return to the closed form representation of $R(\underline{x})$ and $P(\underline{x})$ in order to apply the market optimality conditions (2.8), (2.9). It is for this reason that we directly utilize the completeness of the market and approach the problem in the form of P1 and obtain the optimal mix of securities (or the unit of post-reinsurance wealth) explicitly.

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